

# The constructions of directed strongly regular graph by Algebraic method

Yiqin He, Bicheng Zhang\*, Huabin Cao

*Department of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan, 411105, PR China*

---

## Abstract

The concept of directed strongly regular graphs (DSRG) was introduced by Duval in “A Directed Graph Version of Strongly Regular Graphs” [Journal of Combinatorial Theory, Series A 47(1988)71-100]. Duval also provided several construction methods for directed strongly regular graphs. In this paper, We construct several new families of directed strongly regular graphs which are obtained by using Kronecker matrix product, Semidirect product, Cayley coset graph. At the same time, we give some sufficient and necessary conditions of two special families of Cayley graphs to be DSRG by using group representation theory. At last, we discuss some propositions of in(out)-neighbour set and automorphism group in directed strongly regular graphs.

*Keywords:* Directed strongly regular graphs, Semidirect product, Group representation theory, Cayley graph

---

## 1. Introduction

A *directed strongly regular graph* (DSRG) with parameters  $(n, k, \mu, \lambda, t)$  is a  $k$ -regular directed graph on  $n$  vertices such that every vertex is on  $t$  2-cycles, and the number of paths of length two from a vertex  $x$  to a vertex  $y$  is  $\lambda$  if there is an edge directed from  $x$  to  $y$  and it is  $\mu$  otherwise. A DSRG with  $t = k$  is

---

\*Corresponding author

*Email addresses:* 2014750113@smail.xtu.edu.cn (Yiqin He), zhangbicheng@xtu.edu.cn (Bicheng Zhang), 2014750117@smail.xtu.edu.cn (Huabin Cao)

an (undirected) strongly regular graph (SRG). Duval showed that DSRGs with  $t = 0$  are the doubly regular tournaments. It is therefore usually assumed that  $0 < t < k$ .

Another definition of a directed strongly regular graph, in terms of its adjacency matrix. Let  $D$  be a directed graph with  $n$  vertices. Let  $A = A(D)$  denote the adjacency matrix of  $D$ , and let  $I = I_n$  and  $J = J_n$  denote the  $n \times n$  identity matrix and all-ones matrix, respectively. Then  $D$  is a directed strongly regular graph with parameters  $(n, k, \mu, \lambda, t)$  if and only if (i)  $JA = AJ = kJ$  and (ii)  $A^2 = tI + \lambda A + \mu(J - I - A)$ . Duval gave some propositions of DSRG in [1].

We shall assume that the reader is familiar with standard terminology on directed graphs(see,e.g.,[2]).The vertex set and the arc set of a digraph  $D$  are denote by  $V(D)$  and  $E(D)$ . If  $xy$  is an arc of digraph  $D$ , we say  $x$  *dominates*  $y$  and write  $x \rightarrow y$ . Then the *out-neighbour set*  $N_D^+(x)$  of a vertex  $x$  is the set of vertices dominated by  $x$ , and the *in-neighbour set*  $N_D^-(x)$  of a vertex  $x$  is the set of vertices dominating  $x$ . The number  $d_D^+(v) = |N_D^+(x)|$  and  $d_D^-(v) = |N_D^-(x)|$  are the *outdegree* and *indegree* of  $x$ .

**Proposition 1.** (see [1]) *A directed strongly regular graph with parameters  $(n, k, \mu, \lambda, t)$  with  $0 < t < k$  satisfy*

$$k(k + (\mu - \lambda)) = t + (n - 1)\mu,$$

$$d^2 = (\mu - \lambda)^2 + 4(t - \mu), d | 2k - (\lambda - \mu)(n - 1),$$

$$\frac{2k - (\lambda - \mu)(n - 1)}{d} \equiv n - 1 \pmod{2}, \left| \frac{2k - (\lambda - \mu)(n - 1)}{d} \right| \leq n - 1,$$

where  $d$  is a positive integer, and

$$0 \leq \lambda < t, 0 < \mu \leq t, -2(k - t - 1) \leq \mu - \lambda \leq 2(k - t).$$

**Proposition 2.** (see [1]) *A directed strongly regular graph with parameters  $(n, k, \mu, \lambda, t)$  has three distinct integer eigenvalues*

$$k > \rho = \frac{1}{2}(-(\mu - \lambda) + d) > \sigma = \frac{1}{2}(-(\mu - \lambda) - d),$$

The multiplicities are

$$1, r = -\frac{k + \sigma(n-1)}{\rho - \sigma}, s = \frac{k + \rho(n-1)}{\rho - \sigma}, \quad (1)$$

respectively.

**Proposition 3.** (see [1]) If  $G$  is a DSRG with parameters  $(n, k, \mu, \lambda, t)$ , then the complementary  $G'$  is also a DSRG with parameters  $(n', k', \mu', \lambda', t')$ , where  $k' = (n - 2k) + (k - 1)$ ,  $\lambda' = (n - 2k) + (\mu - 2)$ ,  $t' = (n - 2k) + (t - 1)$ ,  $\mu' = (n - 2k) + \lambda$ .

We note that, although two DSRGs have the same parameters, they may not be isomorphic.

There are many construction methods for DSRG. Duval described some methods including constructions using quadratic residue, block construction of permutation matrices and the Kronecker product [1]. In addition, some of known constructions use combinatorial block designs [3], coherent algebras[3], semidirect product [4], finite geometries [5, 3], matrices[6], block matrices[7], finite incidence structures [8, 9] and Cayley graph[10, 3] or generalized Cayley graph [11].

In section 2, we construct 3 families of DSRGs by using Kronecker matrix product. We construct DSRGs with parameters  $(4n^2, 2n^2 - 2, n^2 - 1, n^2 - 3, n^2 - 1)$ ,  $(4n^2, 2n^2, n^2 + 1, n^2 - 1, n^2 + 1)$  for odd integer  $n$  and  $(4n^2, 2n^2, n^2 + 4, n^2 - 4, n^2 + 4)$  for integer  $4|n$ .

In section 3, by using *semidirect product*, we construct 4 families of DSRGs with parameters:

- (1)  $(pn, vn, \frac{n}{p-1}v^2, \frac{n}{p-1}v(v-1), \frac{n}{p-1}v^2)$  for  $1 \leq v \leq p-2$ ,
- (2)  $(pn, n(v+1)-1, \frac{n}{p-1}v(v+1), n-2+\frac{n}{p-1}v^2, n-1+\frac{n}{p-1}v^2)$  for  $1 \leq v \leq p-2$ ;
- (3)  $(mq, m+q-2, \frac{m-1}{q}+1, \frac{m-1}{q}+q-2, \frac{m-1}{q}+q-1)$ ;
- (4)  $(p^2n, p(p-1)n, n((p-1)^2+1), n\frac{(p-1)^3-1}{p-1}, n((p-1)^2+1))$ .

In section 4, we give a sufficient and necessary condition of Cayley coset graph to be DSRG with parameters  $(n, k, \mu, \lambda, t)$  in terms of the group ring.

In section 5, we focus on the Cayley graph  $\mathcal{C}(G, S)$  where the group  $G$  is two semidirect product of two cyclic group. By using group representation theory, we

can determine the eigenvalues and minimum polynomial of the adjacent matrix of Cayley graph, so we can give some sufficient and necessary condition of Cayley graphs  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m)$  and  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, (A' + e_A) \times C_m \setminus e_A)$  to be DSRG, where  $A'$  is a proper subset of  $C_n \setminus e$  ( $e$  is the identity element of group  $C_n$ ). This suggests that the group representation theory can be used to investigate DSRG.

In section 6, we make a discussion of the vertices which have the same out-neighbour set (in-neighbours set) in DSRG. Meanwhile, we also give an upper bound of the order of automorphism group of directed strongly regular Cayley graph.

## 2. Constructions of DSRG by using Kronecker matrix product

In this section, we construct directed strongly regular graphs from known DSRGs in terms of Kronecker matrix product. Duval [1] observed that if  $t = \mu$  and  $A$  satisfies above equations (i) and (ii), then so does  $A \otimes J_m$  for every positive integer  $m$ ; and so, we have:

**Proposition 4.** *(see [1]) If there exists a directed strongly regular graph with parameters  $(n, k, \mu, \lambda, t)$  with  $t = \mu$ , then for each positive integer  $m$  there exists a directed strongly regular graph with parameters  $(mn, mk, m\mu, m\lambda, mt)$*

Duval [1] also observed that if there exists a DSRG with parameters  $(n, k, \mu, \lambda, t)$  with  $t = \lambda + 1$ , then digraph corresponding to adjacency matrix  $A \otimes J_m + I_n \otimes (J_m - I_m)$  is a DSRG with parameters  $(mn, mk + 1 - 1, m\mu, m(t + 1) - 2, m(t + 1) - 1)$ .

**Definition 1.** Let  $A = (a_{ij})_{m \times n}$  and  $B = (b_{ij})_{t \times s}$  are two matrices over number field  $\mathbb{K}$ , then the Kronecker matrix product of  $A$  and  $B$  is defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Let  $D(A)$  denote with the digraph respect to adjacent matrix  $A$ .

**Theorem 1.** *Let  $D$  be a directed strongly regular graph with the parameters  $(n, k, \mu, \lambda, t)$  satisfies  $t = \mu$  and  $4k = n + 2\lambda + 2\mu$ . Let  $A = A(D)$  be the adjacent matrix of  $D$ , and  $B = (J - A) \otimes A + A \otimes (J - A)$ . Then  $D(B)$  is a directed strongly regular graph with the parameters*

$$(n^2, 2k(n - k), 2(k^2 - 2\mu\lambda), 2(k^2 - \lambda^2 - \mu^2), 2(k^2 - 2\mu\lambda)).$$

*Proof.* Note that  $J^2 = nJ$ , and  $JA = AJ = kJ$ . Thus by a simple computation, we can obtain that

$$\begin{aligned} (J \otimes J)B &= J(J - A) \otimes JA + JA \otimes J(J - A) = 2k(n - k)J \otimes J, \\ B(J \otimes J) &= (J - A)J \otimes AJ + AJ \otimes (J - A)J = 2k(n - k)J \otimes J, \\ B^2 &= (J - A)^2 \otimes A^2 + A^2 \otimes (J - A)^2 + 2(J - A)A \otimes A(J - A). \end{aligned} \quad (2)$$

Note that

$$\begin{aligned} A^2 &= tI + \lambda A + \mu(J - I - A) = \mu J + (\lambda - \mu)A, \\ (J - A)A &= A(J - A) = (k - \mu)J - (\lambda - \mu)A, \\ (J - A)^2 &= J^2 + A^2 - 2kJ = (n - 2k + \mu)J + (\lambda - \mu)A. \end{aligned}$$

Thus, a simple calculation shows that

$$\begin{aligned} (J - A)^2 \otimes A^2 &= ((n - 2k + \mu)J + (\lambda - \mu)A) \otimes (\mu J + (\lambda - \mu)A) \\ &= \mu(n - 2k + \mu)J \otimes J + (n - 2k + \mu)(\lambda - \mu)J \otimes A \\ &\quad + \mu(\lambda - \mu)A \otimes J + (\lambda - \mu)^2 A \otimes A, \end{aligned} \quad (3)$$

$$\begin{aligned} A^2 \otimes (J - A)^2 &= (\mu J + (\lambda - \mu)A) \otimes ((n - 2k + \mu)J + (\lambda - \mu)A) \\ &= \mu(n - 2k + \mu)J \otimes J + (n - 2k + \mu)(\lambda - \mu)A \otimes J \\ &\quad + \mu(\lambda - \mu)J \otimes A + (\lambda - \mu)^2 A \otimes A, \end{aligned} \quad (4)$$

and

$$2(J - A)A \otimes A(J - A) = 2((k - \mu)J - (\lambda - \mu)A) \otimes ((k - \mu)J - (\lambda - \mu)A)$$

$$= 2(k - \mu)^2 J \otimes J + 2(\lambda - \mu)^2 A \otimes A - 2(\lambda - \mu)(k - \mu)(J \otimes A + A \otimes J). \quad (5)$$

We note that  $2\mu(n - 2k + \mu) + 2(k - \mu)^2 = 2\mu(2k - 2\lambda - \mu) + 2(k - \mu)^2 = 2(k^2 - 2\mu\lambda)$ .

Thus from (2), (3), (4), (5),

$$\begin{aligned} B^2 &= (2\mu(n - 2k + \mu) + 2(k - 2\mu)^2)J \otimes J + 4(\lambda - \mu)^2 A \otimes A \\ &\quad + (\lambda - \mu)(n - 2k + 2\mu - 2k + 2\mu)(J \otimes A + A \otimes J) \\ &= 2(k^2 - 2\mu\lambda)J \otimes J + (\lambda - \mu)(2n - 8k + 4\mu + 4\lambda)A \otimes A \\ &\quad + (\lambda - \mu)(n - 4k + 4\mu)((J - A) \otimes A + A \otimes (J - A)) \\ &= 2(k^2 - 2\mu\lambda)J \otimes J + (\lambda - \mu)(n - 4k + 4\mu)B. \end{aligned}$$

Hence  $D(B)$  is a DSRG, let the parameters of  $D(B)$  be  $(n_1, k_1, \mu_1, \lambda_1, t_1)$ , then  $n_1 = n^2, k_1 = 2k(n - k), \mu_1 = t_1 = 2(k^2 - 2\mu\lambda), \lambda_1 = 2(k^2 - 2\mu\lambda) + (\lambda - \mu)(n - 4k + 4\mu) = 2(k^2 - 2\mu\lambda) - 2(\lambda - \mu)^2 = 2(k^2 - \lambda^2 - \mu^2)$ . This completes the proof.  $\square$

**Lemma 2.** (see [12],[13]) If  $(n, k, \mu, \lambda, t)$  are the parameters of a DSRG with  $t = \mu$  and rank  $r$  and with  $\frac{k}{n} = \frac{a}{b}$ , where  $a$  and  $b$  are relatively prime integers, then

$$(n, k, \mu, \lambda, t) = \left( \frac{(r-1)b^2}{c}m, \frac{(r-1)ab}{c}m, \frac{ra^2}{c}m, \frac{(ar-b)a}{c}m, \frac{ra^2}{c}m \right)$$

for some positive integer  $m$ , where  $c$  is the greatest common divisor

$$c = \gcd(ab, ra^2, (r-1)b^2).$$

We now give the possible parameters of DSRGs with  $t = \mu$ , and  $4k = n + 2\lambda + 2\mu$ .

**Theorem 3.** If  $(n, k, \mu, \lambda, t)$  are the parameters of a DSRG with  $t = \mu$ ,  $4k = n + 2\lambda + 2\mu$ , then for positive integer  $m$ , the possible parameters have just 4 classes:

- (1)  $(4(r-1)m, 2(r-1)m, rm, (r-2)m, rm)$  for odd integer  $r$ ;
- (2)  $(2(r-1)m, (r-1)m, \frac{r}{2}m, (\frac{r}{2}-1)m, \frac{r}{2}m)$  for even integer  $r$ ;
- (3)  $(2rm, (r-1)m, \frac{r-1}{2}m, \frac{r-3}{2}m, \frac{r-1}{2}m)$  for odd integer  $r$ ;
- (4)  $(4rm, 2(r-1)m, (r-1)m, (r-3)m, (r-1)m)$  for even integer  $r$ .

*Proof.* From Lemma 2, let  $n = \frac{(r-1)b^2}{c}m$ ,  $k = \frac{(r-1)ab}{c}m$ ,  $\mu = t = \frac{ra^2}{c}m$ ,  $\lambda = \frac{(ar-b)a}{c}m$ . Then  $4k = n + 2\lambda + 2\mu$  implies that

$$4\frac{(r-1)ab}{c}m = \frac{(r-1)b^2}{c}m + 2\frac{ra^2}{c}m + 2\frac{(ar-b)a}{c}m,$$

then  $4(r-1)ab = (r-1)b^2 + 2ra^2 + 2(ar-b)a$ . Thus

$$r(b-2a)^2 = b(b-2a). \quad (6)$$

**Case 1.** If  $b-2a = 0$ , then  $c = \gcd(2a^2, ra^2, 4(r-1)a^2) = a^2 \gcd(2, r)$  and  $n = \frac{4(r-1)}{(2,r)}m$ ,  $k = \frac{2(r-1)}{(2,r)}m$ ,  $\mu = \frac{r}{(2,r)}m$ ,  $\lambda = \frac{r-2}{(2,r)}m$ ,  $t = \frac{r}{(2,r)}m$ . If  $2 \nmid r$ , then  $(n, k, \mu, \lambda, t) = (4(r-1)m, 2(r-1)m, rm, (r-2)m, rm)$ . If  $2 \mid r$ , then  $(n, k, \mu, \lambda, t) = (2(r-1)m, (r-1)m, \frac{r}{2}m, (\frac{r}{2}-1)m, \frac{r}{2}m)$ .

**Case 2.** If  $b-2a \neq 0$ , then  $r(b-2a) = b$ ,  $b(r-1) = 2ar$ ,  $b \mid 2ar$ . Since  $\gcd(a, b) = 1$ , we can obtain that  $b \mid 2r$ .

If  $2 \nmid b$ , then  $b \mid r$ . Hence  $b-2a = 1$ ,  $r = b = 2a+1$  and  $c = \gcd(a(2a+1), (2a+1)a^2, 2a(2a+1)^2) = a(2a+1) = \frac{r(r-1)}{2}$ . Thus  $(n, k, \mu, \lambda, t) = (\frac{(r-1)b^2}{c}m, \frac{(r-1)ab}{c}m, \frac{ra^2}{c}m, \frac{(ar-b)a}{c}m, \frac{ra^2}{c}m) = (2rm, (r-1)m, \frac{r-1}{2}m, \frac{r-3}{2}m, \frac{r-1}{2}m)$ .

If  $2 \mid b$ , let  $b = 2b'$ , then  $b' \mid r$ . By equation (6), we have  $r(b'-a) = b'$ . Thus  $b'-a = 1$ ,  $b = 2(a+1)$  and  $r = a+1$ . Then  $c = \gcd(2a(a+1), (a+1)a^2, 4a(a+1)^2) = a(a+1)\gcd(2, a, 4(a+1)) = a(a+1) = r(r-1)$ , so  $(n, k, \mu, \lambda, t) = (4rm, 2(r-1)m, (r-1)m, (r-3)m, (r-1)m)$ .  $\square$

**Remark 1.** These possible parameters can be rewritten as

(R1)  $(2nm, nm, (\frac{n}{2}+1)m, (\frac{n}{2}-1)m, (\frac{n}{2}+1)m)$  for integer  $4 \mid n$  ( $2(r-1)$  is replaced by  $n$ );

(R2)  $(2nm, nm, \frac{n+1}{2}m, \frac{n-1}{2}m, \frac{n+1}{2}m)$  for odd integer  $n$  ( $r-1$  is replaced by  $n$ );

(R3)  $(2nm, (n-1)m, \frac{n-1}{2}m, \frac{n-3}{2}m, \frac{n-1}{2}m)$  for odd integer  $n$  ( $r$  is replaced by  $n$ );

(R4)  $(2nm, (n-2)m, (\frac{n}{2}-1)m, (\frac{n}{2}-3)m, (\frac{n}{2}-1)m)$  integer  $4 \mid n$  ( $2r$  is replaced by  $n$ ).

For odd  $n$ , the DSRG with parameters  $(2n, n-1, \frac{n-1}{2}, \frac{n-3}{2}, \frac{n-1}{2})$  corresponding to case (R3) in Remark 1 is realizable which was constructed in [6] by using Cayley graph. Then from the Proposition 3, the complementary graph of  $(2n, n-$

$1, \frac{n-1}{2}, \frac{n-3}{2}, \frac{n-1}{2}$ )-DSRG is a DSRG with parameters  $(2n, n, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n+1}{2})$  which corresponding to case (R2).

For even  $n$ , the DSRG with parameters  $(2n, n-1, \frac{n}{2}-1, \frac{n}{2}-1, \frac{n}{2})$  was constructed in [10]. Then for integer  $4|n$ , the complementary graph of it is a DSRG with parameters  $(2n, n, \frac{n}{2}+1, \frac{n}{2}-1, \frac{n}{2}+1)$  which corresponding to case (R1). Thus we have following 3 Theorems.

**Theorem 4.** *For odd integer  $n$ , there exist directed strongly regular graph with the parameters  $(4n^2, 2n^2-2, n^2-1, n^2-3, n^2-1)$ .*

*Proof.* A DSRG with parameters  $(2n, n-1, \frac{n-1}{2}, \frac{n-3}{2}, \frac{n-1}{2})$  was constructed, then Theorem 4 follows from Theorem 1.  $\square$

**Theorem 5.** *For odd integer  $n$ , there exist directed strongly regular graph with the parameters  $(4n^2, 2n^2, n^2+1, n^2-1, n^2+1)$ .*

*Proof.* A DSRG with parameters  $(2n, n, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n+1}{2})$  was constructed, then Theorem 5 follows from Theorem 1.  $\square$

**Theorem 6.** *For integer  $4|n$ , there exist directed strongly regular graph with the parameters  $(4n^2, 2n^2, n^2+4, n^2-4, n^2+4)$ .*

*Proof.* A DSRG with parameters  $(2n, n, \frac{n}{2}+1, \frac{n}{2}-1, \frac{n}{2}+1)$  was constructed, then Theorem 6 follows from Theorem 1.  $\square$

### 3. Constructions of DSRG(Cayley graphs) by using Semidirect product

In [4] Duval, Art M., and Dmitri Iourinski constructed a new infinite family of directed strongly regular graphs, as Cayley graphs of certain semidirect product groups, these results generalizes an earlier construction of Klin, Munemasa, Muzychuk, and Zieschang on some dihedral groups.

In this section, using classical number theory, we also construct some new families directed strongly regular Cayley graphs  $\mathcal{C}(G, S)$ , where groups  $G$  is semidirect product of two groups.



At first, we give some basic definitions of *Cayley graph*, *group ring*, Semidirect product and *primitive root*. Let  $e_G$  or simply  $e$  to be the identity element of  $G$ . The following definitions are the same with Duval [4].

**Definition 2.** Let  $G$  be a finite group and  $S \subset G \setminus e$  (Remove identity element  $e$  from  $G$ ). The Cayley graph of  $G$  generated by  $S$ , denoted by  $\mathcal{C}(G, S)$ , is the digraph  $H$  such that  $V(H) = G$  and  $x \rightarrow y$  if and only if  $x^{-1}y \in S$ , for  $\forall x, y \in G$ .

**Definition 3.** For any finite group  $G$ , the group ring  $\mathbb{Z}[G]$  is defined as the set of all formal sums of elements of  $G$ , with coefficients from  $\mathbb{Z}$ . i.e.

$$\mathbb{Z}[G] = \left\{ \sum_{g \in G} r_g g \mid r_g \in \mathbb{Z}, r_g \neq 0 \text{ for finite } g \right\}.$$

The operations  $+$  and  $\cdot$  on  $\mathbb{Z}[G]$  are given by

$$\begin{aligned} \sum_{g \in G} r_g g + \sum_{g \in G} s_g g &= \sum_{g \in G} (r_g + s_g) g, \\ \left( \sum_{g \in G} r_g g \right) \cdot \left( \sum_{g \in G} s_g g \right) &= \left( \sum_{g \in G} t_g g \right), t_g = \sum_{g'g''=g} r_{g'} s_{g''} \end{aligned}$$

For any subset  $X$  of  $G$ , Let  $\overline{X}$  denote the the element of the group ring  $\mathbb{Z}[G]$  that is the sum of all elements of  $X$ . i.e.

$$\overline{X} = \sum_{x \in X} x$$

The Lemma below allows us to express a sufficient and necessary condition for a Cayley graph to be directed strongly regular graph in terms of the group ring.

**Lemma 7.** *The Cayley graph  $\mathcal{C}(G, S)$  of  $G$  with respect to  $S$  is a directed strongly regular graph with parameters  $(n, k, \mu, \lambda, t)$  if and only if  $|G| = n$ ,  $|S| = k$ , and*

$$\overline{S}^2 = te + \lambda \overline{S} + \mu(\overline{G} - e - \overline{S}). \quad (7)$$

**Definition 4.** Let  $\theta : B \rightarrow \text{Aut } A$  be an action of a group  $B$  on another group  $A$ : Let  $A \rtimes_{\theta} B$  be the direct product set of  $A$  and  $B$ ; i.e., set of pairs  $(a, b)$  operation for the product of two elements

$$(a, b)(a', b') = (a[\theta(b)(a')], bb')$$

Then  $A \rtimes_{\theta} B$  is a group of order  $|A||B|$ , This group is called the semidirect product of  $A$  and  $B$  with respect to the action  $\theta$ .

From the definition of semidirect product of  $A$  and  $B$ , we can obtain that  $A$  is isomorphic to  $A \times e_B$ , i.e.,  $A \cong A \times e_B$ , and  $B$  is isomorphic to  $e_A \times B$ , i.e.,  $B \cong e_A \times B$ . Then we can equate  $a$  with  $(a, e_B)$  for  $a \in A$ , and  $b$  with  $(e_A, b)$  for  $b \in B$ , and it is easy to verify that  $ab = (a, e_B)(e_A, b) = (a, b)$ ,  $ba = (e_A, b)(a, e_B) = (a, b)(a', b') = ([\theta(b)(a)], b) = [\theta(b)(a)]b$ ,  $e_A = (e_A, e_B) = e_B = e$  ( $e$  is the identity element of  $A \rtimes_{\theta} B$ ).

**Definition 5.** For positive integer  $s$ , which coprime with  $n$ , we denote  $\delta_n(s)$  with the the least positive integer  $k$  such that  $s^k \equiv 1 \pmod{n}$ . If  $\delta_n(s)$  is equal to the totient of  $n$ , the number of positive integers that are both less  $n$  and coprime to  $n$ , i.e.  $\delta_n(s) = \varphi(n)$ , then we say that  $s$  is a primitive root of modulo  $n$ .

From the classical number theory, the following result characterizes the existence of primitive root of modulo  $n$ .

**Lemma 8.** *There is a primitive root of modulo  $n$  if and only if  $n = 2$  or  $4$ , or  $n = p^k$  or  $2p^k$  for odd prime  $p$  and  $k \in \mathbb{N}$ .*

Let  $C_n = \langle x \rangle$  be multiplicative cyclic groups of orders  $n$  with generator  $x$ .

### 3.1. Cayley graph $\mathcal{C}(C_p \rtimes_{\theta(m)} C_n, S)$ to be DSRG

Let  $A = C_p = \langle a \rangle$  be multiplicative cyclic groups of odd prime orders  $p$ ,  $B = C_n = \langle x \rangle$  be another multiplicative cyclic groups of orders  $n$ .

Let  $m$  be an integer such that  $(m, p) = 1$ ,  $m \not\equiv 1 \pmod{p}$ ,  $m^n \equiv 1 \pmod{p}$ . The map  $\beta_m \in \text{Aut } C_p$  given by  $\beta_m : a^i \rightarrow a^{mi}$  is an automorphism, and the

map  $\theta(m): B \rightarrow \text{Aut } C_p$  given by  $\theta(m)(x^u) = \beta_m^u$  is a homomorphism. Then  $A \rtimes_{\theta(m)} B = \langle a, x | a^p = x^n = e, xa = a^m x \rangle$  is a group of order  $pn$ .

Let  $H$  be a proper subset of  $\{1, 2, \dots, p-1\}$ . We denote  $v$  with the cardinalty of  $H$ , i.e.  $|H| = v$ .

3.1.1.  $(pn, vn, \frac{n}{p-1}v^2, \frac{n}{p-1}v(v-1), \frac{n}{p-1}v^2)$ -DSRG for  $1 \leq v \leq p-2$

**Theorem 9.** Let  $A = C_p$ ,  $p$  be an odd prime,  $m$  be a primitive root of module  $p$ ,  $n$  be an integer such that  $p-1 \mid n$ . Let  $B = \langle x | x^n = e_B \rangle$  and  $A' = \{a^l | l \in H\}$ . Then the Cayley graph  $\mathcal{C}(A \rtimes_{\theta(m)} B, A' \times B)$  is a directed strongly regular graph with the parameters

$$(pn, vn, \frac{n}{p-1}v^2, \frac{n}{p-1}v(v-1), \frac{n}{p-1}v^2).$$

*Proof.* Let  $S = A' \times B$ , first we compute  $\overline{B} \overline{A'} \overline{B}$  in the group ring  $\mathbb{Z}[A \rtimes_{\theta(m)} B]$ :

$$\begin{aligned} \overline{B} \overline{A'} \overline{B} &= \left( \sum_{u=0}^{n-1} x^u \right) \left( \sum_{l \in H} a^l \right) \overline{B} \\ &= \sum_{l \in H} \sum_{u=0}^{n-1} [\theta(x^u)(a^l)] x^u \overline{B} \\ &= \sum_{l \in H} \sum_{u=0}^{n-1} \beta_m^u(a^l) \overline{B} \\ &= \sum_{l \in H} \sum_{u=0}^{n-1} a^{lm^u} \overline{B} \\ &= \frac{n|H|}{p-1} \sum_{l=1}^{p-1} a^l \overline{B} \quad . \end{aligned} \tag{8}$$

The last equation follows from  $\gcd(l, p) = 1$  for each  $l \in H$ . Then

$$\begin{aligned} \overline{S}^2 &= (\overline{A' \times B})^2 = \overline{A'} \overline{B} \overline{A'} \overline{B} \\ &= \frac{n|H|}{p-1} \sum_{l' \in H} \sum_{l=1}^{p-1} a^{l'+l} \overline{B} \\ &= \frac{n|H|}{p-1} \sum_{l' \in H} \sum_{\substack{l=0 \\ l \neq l'}}^{p-1} a^l \overline{B} \end{aligned}$$

$$\begin{aligned}
&= \frac{n|H|^2}{p-1} \sum_{l=0}^{p-1} a^l \bar{B} - \frac{n|H|}{p-1} \sum_{l' \in H} a^{l'} \bar{B} \\
&= \frac{nv^2}{p-1} \overline{A \rtimes_{\theta} B} - \frac{nv}{p-1} \bar{S}.
\end{aligned} \tag{9}$$

Thus

$$\bar{S}^2 = \frac{nv^2}{p-1} e + \left( \frac{nv^2}{p-1} - \frac{nv}{p-1} \right) \bar{S} + \frac{nv^2}{p-1} (\overline{A \rtimes_{\theta} B} - e - \bar{S}).$$

Hence from Lemma 7, Cayley graph  $\mathcal{C}(A \rtimes_{\theta(m)} B, A' \times B)$  is a directed strongly regular graph with the parameters

$$(pn, vn, \frac{n}{p-1}v^2, \frac{n}{p-1}v(v-1), \frac{n}{p-1}v^2).$$

□

**Remark 2.** Indeed, if  $v = p-1$ , then we can also get  $(\alpha p(p-1), \alpha(p-1)^2, \alpha(p-1)^2, \alpha(p-2)(p-1), \alpha(p-1)^2)$ -DSRG, but it is a SRG in this case.

*3.1.2.  $(pn, n(v+1)-1, \frac{nv(v+1)}{p-1}, n-2+\frac{nv^2}{p-1}, n-1+\frac{nv^2}{p-1})$ -DSRG for  $1 \leq v \leq p-2$*

The subset  $(A' + e_A) \times B \setminus e_A$  of  $A \rtimes_{\theta(m)} B$  means that remove element  $e_A$  from  $(A' + e_A) \times B$ , so we can get  $(A' + e_A) \times B \setminus e_A = \{a^l x^j, a^c | 1 \leq j \leq n-1, l \in H \cup \{0\}, c \in H\}$ ,

**Theorem 10.** Let  $A = C_p = \langle a \rangle$ ,  $p$  be an odd prime,  $m$  be a primitive root of module  $p$ ,  $n$  be an integer such that  $p-1 \mid n$ . Let  $B = C_n = \langle x | x^n = e_B \rangle$ , and  $A' = \{a^l | l \in H\}$ . Then the Cayley graph  $\mathcal{C}(A \rtimes_{\theta(m)} B, (A' + e_A) \times B \setminus e_A)$  is a directed strongly regular graph with the parameters

$$(pn, n(v+1)-1, \frac{nv(v+1)}{p-1}, n-2+\frac{nv^2}{p-1}, n-1+\frac{nv^2}{p-1}).$$

*Proof.* Let  $S = (A' + e_A) \times B \setminus e_A$ , then  $\bar{S} = \sum_{k \in H} a^k \bar{B} + \bar{B} - e_A = \bar{A'} \bar{B} + \bar{B} - e_A$ .

Thus

$$\begin{aligned}
\bar{S}^2 &= (\bar{A'} \bar{B} + \bar{B} - e_A)^2 \\
&= (\bar{A'} \bar{B})^2 + (\bar{B} - e_A)^2 + (\bar{B} - e_A) \bar{A'} \bar{B} + \bar{A'} \bar{B} (\bar{B} - e_A) \\
&= (\bar{A'} \bar{B})^2 + (|B| - 2) \bar{B} + e_A + \bar{B} \bar{A'} \bar{B} - \bar{A'} \bar{B} + (|B| - 1) \bar{A'} \bar{B}
\end{aligned}$$

$$= (\overline{A'} \overline{B})^2 + e_A + \overline{B} \overline{A'} \overline{B} + (|B| - 2)(\overline{A'} \overline{B} + \overline{B}).$$

It follows from equations (8) and (9) in Theorem 9 that

$$\overline{B} \overline{A'} \overline{B} = \frac{nv}{p-1} \sum_{k=1}^{p-1} a^k \overline{B} = \frac{nv}{p-1} (\overline{A} - e_A) \overline{B}$$

and

$$(\overline{A'} \overline{B})^2 = \frac{nv^2}{p-1} \overline{A \rtimes_{\theta(m)} B} - \frac{nv}{p-1} \overline{A'} \overline{B}.$$

Then  $\overline{S}^2$  becomes

$$\begin{aligned} &= \frac{nv^2}{p-1} \overline{A \rtimes_{\theta(m)} B} - \frac{nv}{p-1} \overline{A'} \overline{B} + \frac{nv}{p-1} (\overline{A} - e_A) \overline{B} + e_A + (|B| - 2)(\overline{S} + e_A) \\ &= \frac{nv^2}{p-1} \overline{A \rtimes_{\theta(m)} B} - \frac{nv}{p-1} (\overline{A'} + e_A) \overline{B} + \frac{nv}{p-1} \overline{A \rtimes_{\theta(m)} B} + (n-2)\overline{S} + (n-1)e_A \\ &= \frac{nv(v+1)}{p-1} \overline{A \rtimes_{\theta(m)} B} - \frac{nv}{p-1} (\overline{S} + e_A) + (n-2)\overline{S} + (n-1)e_A \\ &= \frac{nv(v+1)}{p-1} \overline{A \rtimes_{\theta(m)} B} + (n-2 - \frac{nv}{p-1})\overline{S} + (n-1 - \frac{nv}{p-1})e. \end{aligned}$$

Thus

$$\overline{S}^2 = (n-1 + \frac{nv^2}{p-1})e + (n-2 + \frac{nv^2}{p-1})\overline{S} + \frac{nv(v+1)}{p-1} (\overline{A \rtimes_{\theta(m)} B} - e - \overline{S}).$$

Hence from Lemma 7, Cayley graph  $\mathcal{C}(A \rtimes_{\theta(m)} B, S)$  is a directed strongly regular graph with the parameters

$$(pn, n(v+1) - 1, \frac{nv(v+1)}{p-1}, n-2 + \frac{nv^2}{p-1}, n-1 + \frac{nv^2}{p-1}).$$

□

### 3.2. $(mq, m+q-2, \frac{m-1}{q} + 1, \frac{m-1}{q} + q-2, \frac{m-1}{q} + q-1)$ -DSRG

For  $A$  be a finite group of order  $m$  and some  $\beta \in \text{Aut } A$ , Duval, Art M., and Dmitri Iourinski [4] define a group automorphism  $\beta$  which has the  $q$ -orbit condition if each of its untrivial orbits contains  $q$  elements. i.e. If the group  $\langle \beta \rangle$  (The subgroup of  $\text{Aut } A$  generated by  $\beta$ ) acts on  $A$  naturally, then each untrivial orbit have  $q$  elements. In other words,  $\beta$  satisfies  $\beta^q(a) = a$ , for all  $a \in A$ , and  $\beta^u(a) = a$  implies  $q|u$ , for all  $a \neq e_A$ .

Let  $A$  be a finite group of order  $m$  and some  $\beta \in \text{Aut } A$  has the  $q$ -orbit condition, let  $B$  be the cyclic group of order  $q$  with generator  $b$ , and define  $\theta: B \rightarrow \text{Aut } A$  by  $\theta(b^u) = \beta^u$ . let  $A'$  be a set of representatives of the nontrivial orbits of  $\beta$ . Duval observed that the Cayley graph  $\mathcal{C}(A \rtimes_{\theta} B, A' \times B)$  is a directed strongly regular graph with the parameters

$$(mq, m-1, (m-1)/q, ((m-1)/q)-1, (m-1)/q).$$

A similar result will be given in the next Theorem.

**Theorem 11.** *Let  $A$  be a finite group of order  $m$ . If some  $\beta \in \text{Aut } A$  has the  $q$ -orbit condition, let  $B$  be the cyclic group of order  $q$  with generator  $b$ , and define  $\theta: B \rightarrow \text{Aut } A$  by  $\theta(b^u) = \beta^u$  for  $1 \leq u \leq q$ . Let  $A'$  be a set of representatives of the nontrivial orbits of  $\beta$ . Then the Cayley graph  $\mathcal{C}(A \rtimes_{\theta} B, (A' + e_A) \times B \setminus e_A)$  is a directed strongly regular graph with the parameters*

$$(mq, m+q-2, \frac{m-1}{q}+1, \frac{m-1}{q}+q-2, \frac{m-1}{q}+q-1)$$

*Proof.* We recall that  $\overline{B} \overline{A'} \overline{B} = (\overline{A} - e_A) \overline{B} = \overline{A} \overline{B} - \overline{B}$ ,  $\overline{A'} \overline{B} \overline{A'} \overline{B} = |A'| \overline{A \rtimes_{\theta} B} - \overline{A'} \overline{B}$  from the proof of Theorem 3.3 in [4]. Let  $S = (A' + e_A) \times B \setminus e_A$ , then

$$\begin{aligned} \overline{S}^2 &= (\overline{A'} \overline{B} + \overline{B} - e_A)^2 \\ &= (\overline{A'} \overline{B})^2 + (\overline{B} - e_A)^2 + (\overline{B} - e_A) \overline{A'} \overline{B} + \overline{A'} \overline{B} (\overline{B} - e_A) \\ &= |A'| \overline{A \rtimes_{\theta} B} - \overline{A'} \overline{B} + (|B| - 2) \overline{B} + e_A + (|B| - 1) \overline{A'} \overline{B} \\ &\quad + \overline{A} \overline{B} - \overline{B} - \overline{A'} \overline{B} \\ &= (|A'| + 1) \overline{A \rtimes_{\theta} B} + (|B| - 3) \overline{B} + e_A + (|B| - 3) \overline{A'} \overline{B} \\ &= (|A'| + 1) \overline{A \rtimes_{\theta} B} + (|B| - 3) \overline{S} + (|B| - 2) e. \end{aligned}$$

Thus

$$\overline{S}^2 = (|A'| + 1) (\overline{A \rtimes_{\theta} B} - e - \overline{S}) + (|A'| + |B| - 2) \overline{S} + (|A'| + |B| - 1) e.$$

It now only remains to note that  $|B| = q$ ,  $|A'| = \frac{m-1}{q}$ ,  $|A \rtimes_{\theta} B| = mq$ ,  $|(A' + e_A) \times B \setminus e_A| = (\frac{m-1}{q} + 1)q - 1 = m + q - 2$ . Hence from Lemma 7, Cayley graph  $\mathcal{C}(A \rtimes_{\theta} B, (A' + e_A) \times B \setminus e_A)$  is a DSRG with the parameters

$$(mq, m + q - 2, \frac{m-1}{q} + 1, \frac{m-1}{q} + q - 2, \frac{m-1}{q} + q - 1)$$

□

### 3.3. $(p^2n, p(p-1)n, n((p-1)^2 + 1), n\frac{(p-1)^3-1}{p-1}, n((p-1)^2 + 1))$ -DSRG

In the previous sections, we concentrate on the Cayley graph  $\mathcal{C}(G, S)$ ,  $G$  is a semidirect product of two cyclic group. In this section, we discuss Cayley graph  $\mathcal{C}(G, S)$  with  $G = (C_p \rtimes_{\theta(s)} C_n) \rtimes_{\vartheta} B$ , where  $C_p = \langle a \rangle$ ,  $C_n = \langle x \rangle$ , and  $B = \langle y | y^p = e_B \rangle$  is a cyclic group of order  $p$  with generator  $y$ . Let  $s$  be an integer such that  $(s, p) = 1$ ,  $s \not\equiv 1 \pmod{p}$ ,  $s^n \equiv 1 \pmod{p}$ . The map  $\beta_s \in \text{Aut } \langle a \rangle$  given by  $\beta_s : a^i \rightarrow a^{si}$  is an automorphism, and the homomorphism  $\theta(s)$  from  $\langle x \rangle$  to  $\text{Aut } \langle a \rangle$  is defined by:  $x^{\alpha} \rightarrow \beta_s^{\alpha}$ .

Let  $D_{p,n,s} = C_p \rtimes_{\theta(s)} C_n$ , the inner automorphism  $f(g) \in \text{Aut } D_{p,n,s}$  is defined by  $f(g): \phi \rightarrow g^{-1}\phi g$ , for each  $\phi \in D_{p,n,s}$ . And the map  $\vartheta: B \rightarrow \text{Aut } D_{p,n,s}$  given by  $\vartheta(y^u) = f(a^u)$  is a homomorphism. Thus,  $D_{p,n,s} \rtimes_{\vartheta} B$  is a group of order  $p^2n$ . From the definition, we can obtain  $x^t a^u = a^{us^t} x^t$  easily.

Let  $\mathcal{E}(n)$  denote with the set of positive integers that are both less  $n$  and coprime to  $n$ , i.e.  $\mathcal{E}(n) = \{q | 1 \leq q \leq n-1, (q, n) = 1\}$ , so  $|\mathcal{E}(n)| = \varphi(n)$ .

The next two Lemmas will be used in the proof of Theorem 14.

**Lemma 12.** *Let  $A = C_{p^l} = \langle a \rangle$  and  $A' = \{a^k | k \in \mathcal{E}(p^l)\}$ . Then*

$$\overline{A'}^2 = (p^l - p^{l-1})\overline{A} - p^{l-1}\overline{A'}.$$

*Proof.* We compute  $\overline{A'}^2$  in the group ring  $\mathbb{Z}[C_{p^l}]$ :

$$\begin{aligned} \overline{A'}^2 &= \left( \overline{A} - \sum_{k=1}^{p^{l-1}} a^{pk} \right)^2 \\ &= \overline{A}^2 - \sum_{k=1}^{p^{l-1}} a^{pk} \overline{A} - \overline{A} \sum_{k=1}^{p^{l-1}} a^{pk} + \left( \sum_{k=1}^{p^{l-1}} a^{pk} \right)^2 \end{aligned}$$

$$= (p^l - 2p^{l-1})\bar{A} + \sum_{j=1}^{p^{l-1}} \left( \sum_{k=1}^{p^{l-1}} a^{p(k+j)} \right).$$

For each  $1 \leq j \leq p^{l-1}$ , when  $k$  takes over the complete residue system of module  $p^{l-1}$ , the  $k+j$  also takes over the complete residue system of module  $p^{l-1}$ . Then we can get that  $\sum_{k=1}^{p^{l-1}} a^{p(k+j)} = \sum_{k=1}^{p^{l-1}} a^{pk}$ . Thus we can obtain that

$$\begin{aligned} \bar{A'}^2 &= (p^l - 2p^{l-1})\bar{A} + p^{l-1} \left( \sum_{k=1}^{p^{l-1}} a^{pk} \right) \\ &= (p^l - 2p^{l-1})\bar{A} + p^{l-1}(\bar{A} - \bar{A'}) \\ &= (p^l - p^{l-1})\bar{A} - p^{l-1}\bar{A'}. \end{aligned}$$

□

**Lemma 13.** Let  $H \subseteq \{0, 1, \dots, p-1\}$ ,  $T \subseteq \{0, 1, \dots, n-1\}$ ,  $S = \{a^l x^i | l \in H, i \in T\} \times B$ . Then

$$\bar{S}^2 = \sum_{u=0}^{m-1} \sum_{l' \in H} \sum_{t', i \in T} a^{l' + (l-u+us^i)s^{i'}} x^{i'+i} \bar{B}.$$

*Proof.* First we compute  $\bar{B} \bar{A'} \bar{B}$  in the group ring  $\mathbb{Z}[D_{p,n,s} \rtimes_{\vartheta} B]$ :

$$\begin{aligned} \bar{B} \bar{A'} \bar{B} &= \left( \sum_{u=0}^{m-1} y^u \right) \left( \sum_{l \in H, i \in T} a^l x^i \right) \bar{B} \\ &= \sum_{l \in H, i \in T} \sum_{u=0}^{m-1} y^u (a^l x^i) \bar{B} \\ &= \sum_{l \in H, i \in T} \sum_{u=0}^{m-1} [\vartheta(y^u)(a^l x^i)] y^u \bar{B} \\ &= \sum_{l \in H, i \in T} \sum_{u=0}^{m-1} a^{l-u} x^i a^u \bar{B} \\ &= \sum_{l \in H, i \in T} \sum_{u=0}^{m-1} a^{l-u+us^i} x^i \bar{B}. \end{aligned}$$



Thus

$$\begin{aligned}
\bar{S}^2 &= (\overline{A' \times B})^2 = \overline{A'} \overline{B} \overline{A'} \overline{B} \\
&= \overline{A'} \sum_{l \in H, i \in T} \sum_{u=0}^{m-1} a^{l-u+us^i} x^i \overline{B} \\
&= \sum_{u=0}^{m-1} \sum_{l' \in H} \sum_{i' \in T} a^{l'} x^{i'} a^{l-u+us^i} x^i \overline{B} \\
&= \sum_{u=0}^{m-1} \sum_{l' \in H} \sum_{i' \in T} a^{l' + (l-u+us^i)s^{i'}} x^{i' + i} \overline{B}.
\end{aligned}$$

□

**Theorem 14.** Let  $A = D_{p,n,s} = C_p \rtimes_{\theta(s)} C_n$ ,  $p$  be a prime,  $s$  be a primitive root of module  $p$ ,  $n$  be an integer such that  $p-1 \mid n$ .  $B = \langle y | y^p = e_B \rangle$ ,  $A' = \{a^l x^i | l \in \mathcal{E}(p), i \in \{0, 1, \dots, n-1\}\}$ . Then the Cayley graph  $\mathcal{C}(A \rtimes_{\theta} B, A' \times B)$  is a directed strongly regular graph with the parameters

$$(p^2 n, p(p-1)n, n((p-1)^2 + 1), n \frac{(p-1)^3 - 1}{p-1}, n((p-1)^2 + 1)).$$

*Proof.* Let  $S = A' \times B$ , then from Lemma 13 and Lemma 12 for  $l = 1$ , we have

$$\begin{aligned}
\bar{S}^2 &= \sum_{u=0}^{p-1} \sum_{l' \in \mathcal{E}(p)} \sum_{i', i=0}^{n-1} a^{l' + (l-u+us^i)s^{i'}} x^{i' + i} \overline{B} \\
&= \sum_{i', i=0}^{n-1} \sum_{l' \in \mathcal{E}(p)} a^{l' + ls^{i'}} \sum_{u=0}^{p-1} a^{u(s^i - 1)s^{i'}} x^{i' + i} \overline{B} \text{ (Lemma 12)} \\
&= \sum_{i', i=0}^{n-1} \bar{X} \sum_{u=0}^{p-1} a^{u(s^i - 1)s^{i'}} x^{i' + i} \overline{B} \\
&= \sum_{\substack{i', i=0 \\ s^i \not\equiv 1 \pmod{p}}}^{n-1} \bar{X} \sum_{u=0}^{p-1} a^{u(s^i - 1)s^{i'}} x^{i' + i} \overline{B} \\
&+ p \sum_{\substack{i', i=0 \\ s^i \equiv 1 \pmod{p}}}^{n-1} \bar{X} x^{i' + i} \overline{B} \triangleq \Delta_1 + \Delta_2,
\end{aligned}$$

where  $\overline{X} = (p-1) \sum_{l=0}^{p-1} a^l - \sum_{l \in \mathcal{E}(p)} a^l$ . We note that  $\gcd((s^i - 1)s^{i'}, p) = 1$  for each  $i$  satisfies  $s^i \not\equiv 1 \pmod{p}$ . Then

$$\begin{aligned}
\Delta_1 &= \sum_{\substack{i', i=0 \\ s^i \not\equiv 1 \pmod{p}}}^{n-1} \left( (p-1) \sum_{l=0}^{p-1} a^l - \sum_{l \in \mathcal{E}(p)} a^l \right) \sum_{u=0}^{p-1} a^{u(s^i-1)s^{i'}} a^{i'+i} \overline{B} \\
&= \sum_{\substack{i', i=0 \\ s^i \not\equiv 1 \pmod{p}}}^{n-1} \left( (p-1) \sum_{l=0}^{p-1} a^l - \sum_{l \in \mathcal{E}(p)} a^l \right) \left( \sum_{u=0}^{p-1} a^u \right) x^{i'+i} \overline{B} \\
&= (p(p-1) - \varphi(p)) \sum_{\substack{i', i=0 \\ s^i \not\equiv 1 \pmod{p}}}^{n-1} \left( \sum_{u=0}^{p-1} a^u \right) x^{i'+i} \overline{B} \\
&= v(p-1)^2 \sum_{i'=0}^{n-1} \left( \sum_{u=0}^{p-1} a^u \right) x^{i'} \overline{B} = v(p-1)^2 \overline{A \rtimes_{\vartheta} B},
\end{aligned}$$

where  $v = |\{i | s^i \not\equiv 1 \pmod{p}, i \in \{0, 1, \dots, n-1\}\}| = n - \frac{n}{p-1}$ . And

$$\begin{aligned}
\Delta_2 &= p \sum_{\substack{i', i=0 \\ s^i \equiv 1 \pmod{p}}}^{n-1} \left( (p-1) \sum_{l=0}^{p-1} a^l - \sum_{l \in \mathcal{E}(p)} a^l \right) x^{i'+i} \overline{B} \\
&= p(n-v) \sum_{i'=0}^{n-1} \left( (p-1) \sum_{l=0}^{p-1} a^l - \sum_{l \in \mathcal{E}(p)} a^l \right) x^{i'} \overline{B} \\
&= p(n-v)(p-1) \overline{A \rtimes_{\vartheta} B} - p(n-v) \overline{S}.
\end{aligned}$$

Thus

$$\overline{S}^2 = (v(p-1)^2 + p(n-v)(p-1)) \overline{A \rtimes_{\vartheta} B} - p(n-v) \overline{S}.$$

We note that  $v(p-1)^2 + p(n-v)(p-1) = n(p-1)(p-2) + pn = n(p^2 - 2p + 2) = n((p-1)^2 + 1)$ , so

$$\overline{S}^2 = (n((p-1)^2 + 1) - p(n-v))(\overline{S} + e) + n((p-1)^2 + 1)(\overline{A \rtimes_{\vartheta} B} - e - \overline{S}).$$

It now only remains to note that  $|A \rtimes_{\vartheta} B| = p^2 n$ ,  $|A' \times B| = p\varphi(p)n = p(p-1)n$  and  $v(p-1)^2 + p(n-v)(p-1) - p(n-v) = n \frac{(p-1)^3 - 1}{p-1}$ . Thus

Cayley graph  $\mathcal{C}(A \rtimes_{\vartheta} B, A' \times B)$  is a directed strongly regular graph with the parameters

$$(p^2n, p(p-1)n, n((p-1)^2 + 1), n\frac{(p-1)^3 - 1}{p-1}, n((p-1)^2 + 1)).$$

□

#### 4. Constructions of DSRG by using Cayley coset graph

For a group  $G$  and a subgroup  $H \leq G$ , denote by  $[G : H]$  the set of left cosets of  $H$  in  $G$ , that is

$$[G : H] = \{xH | x \in G\}.$$

$|G : H|$  is the index of  $H$  in  $G$ . For any subset  $S \subset G$ , we may define a digraph on  $[G : H]$  as follows:

**Definition 6.** Let  $G$  be a group,  $H$  a subgroup of  $G$ , and  $S$  a subset of  $G$ . define the Cayley coset graph of  $G$  with respect to  $H$  and  $S$  to be the directed graph with vertex set  $[G : H]$  and such that, for any  $xH, yH \in V$ ,  $xH$  is connected to  $yH$  if and only if  $x^{-1}y \in HSH$ , and denote the digraph by  $\mathcal{C}(G, H, HSH)$ .

We note that if  $H = 1$ , the Cayley coset graph is a Cayley graph. In this section we just give a sufficient and necessary condition for a Cayley coset graph to be DSRG in terms of the group ring.

**Lemma 15.** *The number of paths of length 2 from  $xH$  to  $yH$  in  $\mathcal{C}(G, H, HSH)$  equals the coefficient of  $x^{-1}y$  in  $\frac{1}{|H|} \overline{HSH}^2$ .*

*Proof.* The coefficient of  $x^{-1}y$  in  $\overline{HSH}^2$  is the number of ordered pairs  $(x_1, x_2) \in HSH \times HSH$  such that  $x_1x_2 = x^{-1}y$ .

Let  $\mathcal{Q}$  be the set of all the ordered pair  $(x_1, x_2) \in HSH \times HSH$  such that  $x_1x_2 = x^{-1}y$ ,  $\mathcal{P}$  be the set of all paths of length 2 from  $xH$  to  $yH$ . We define a map  $\eta : \mathcal{Q} \rightarrow \mathcal{P}$  by:

$$(x_1, x_2) \rightarrow p(x, xx_1, y),$$

where  $p(x, z, y)$  denote with the path  $xH \rightarrow zH \rightarrow yH$  of length 2.

Let  $\eta^{-1}(p)$  denote with the preimage of  $p$ .

At first, we prove that  $\eta^{-1}(p(x, z, y)) = \{(x^{-1}zh, h^{-1}z^{-1}y) | h \in H\}$ .

We can get  $\{(x^{-1}zh, h^{-1}z^{-1}y) | h \in H\} \subset \eta^{-1}(p(x, z, y))$  easily. It now only remains to prove that  $\eta^{-1}(p(x, z, y)) \subset \{(x^{-1}zh, h^{-1}z^{-1}y) | h \in H\}$ . Indeed, for each  $(x_1, x_2) \in \eta^{-1}(xH \rightarrow zH \rightarrow yH)$ , we can obtain that  $xx_1H = zH$ , i.e.  $x_1 \in x^{-1}zH$ , so  $(x_1, x_2) \in \{(x^{-1}zh, h^{-1}z^{-1}y) | h \in H\}$ . so  $\{(x^{-1}zh, h^{-1}z^{-1}y) | h \in H\} = \eta^{-1}(p(x, z, y))$  and  $|\eta^{-1}(p(x, z, y))| = |H|$ .

Secondly, the map  $\eta$  is a surjection obviously. Thus

$$|\mathcal{Q}| = \sum_{p(x,z,y) \in \mathcal{P}} |\eta^{-1}(p(x, z, y))| = |H||\mathcal{P}|$$

Then the result is now immediate.  $\square$

**Theorem 16.** *The Cayley graph  $\mathcal{C}(G; H; HSH)$  is a directed strongly regular graph with parameters  $(n, k, \mu, \lambda, t)$  if and only if  $|G : H| = n$ ,  $\frac{|HSH|}{|H|} = k$ , and*

$$\frac{1}{|H|} \overline{HSH}^2 = te + \lambda \overline{HSH} + \mu(\overline{G} - e - \overline{HSH}).$$

*Proof.* Suppose  $\frac{1}{|H|} \overline{HSH}^2 = te + \lambda \overline{HSH} + \mu(\overline{G} - e - \overline{HSH})$ , By Lemma 15 then, the number of paths of length 2 from  $xH$  to  $yH$  is:  $t$  if  $x^{-1}y = e$ ; i.e.  $xH = yH$ ;  $\lambda$  if  $x^{-1}y \in HSH$ , i.e.  $xH \rightarrow yH$ ; and  $\mu$  otherwise. Thus, we can get  $\mathcal{C}(G, H, HSH)$  is a directed strongly regular graph. The reverse direction is a direct consequence of Lemma 15.  $\square$

**Remark 3.** if  $H = 1$ , we can get Lemma 7 from Theorem 16 easily.

## 5. Sufficient and necessary condition of Cayley graph $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, S)$ to be DSRG

In this section, we focus on the sufficient and necessary conditions of Cayley graphs  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m)$  and  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, (A' + e_A) \times C_m \setminus e_A)$  to be DSRG, where  $A'$  is a proper subet of  $C_n \setminus e$ . These sufficient and necessary

conditions can be used to verify Cayley graphs which constructed in section 3.1, 3.2, and [4],[10] are DSRGs.

Let  $x$  be a generator for  $C_n$  and  $y$  a generator for  $C_m$ . Let  $k$  be an integer such that  $(k, n) = 1$ ,  $k \not\equiv 1 \pmod{n}$ ,  $k^m \equiv 1 \pmod{n}$ . Then the map  $\beta_k \in \text{Aut } C_n$  given by  $\beta_k : x^i \rightarrow x^{ki}$  is an automorphism, and the map  $\theta(k) : C_m \rightarrow \text{Aut } C_n$  given by  $\theta(k)(y^u) = \beta_k^u$  is a homomorphism.  $A' = \{x^a | a \in H\}$ , where  $H$  is a proper subset of  $\{1, 2, \dots, n-1\}$  with  $|H| = v$ .

### 5.1. A sufficient and necessary condition of Cayley graph

$\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m)$  to be DSRG

At first, we give some basic definitions of group representation theory.

Let  $V$  be a vector space over the field  $\mathbb{C}$  of complex numbers and let  $GL(V)$  be the group of isomorphisms of  $V$  onto itself. Let  $GL_n(\mathbb{C})$  be the general linear group consisting of all invertible matrices of order  $n$ . A representation of  $G$  in  $V$  is a homomorphism from the group  $G$  into the group  $GL(V)$ . Let  $\rho$  and  $\rho'$  be two representations of the same group  $G$  in vector spaces  $V$  and  $V'$ , these representations are said to be isomorphic if there exists a linear isomorphism  $\omega : V \rightarrow V'$ , which satisfies the identity

$$\omega \circ \psi(g) = \varphi(g) \circ \omega$$

for all  $g \in G$ .

Let  $\rho$  be a representations of group  $G$  in vector spaces  $V$  with  $\dim_{\mathbb{C}} V = n$ . Choose a basis for  $V$ , and let  $T : V \rightarrow \mathbb{C}^n$  be the isomorphism taking coordinates with respect to this basis. Then setting  $\varphi_g = T\rho_g T^{-1}$ , for  $g \in G$ , yields a representation  $\varphi : G \rightarrow GL_n(\mathbb{C})$  isomorphic to  $\rho$ . So, if  $\rho$  and  $\rho'$  are given in matrix form by  $\varphi_g$  and  $\varphi'_g$  respectively, then  $\rho$  and  $\rho'$  are isomorphic means that there exists an invertible matrix  $H$  such that for all  $g \in G$

$$H\varphi_g = \varphi'_g H \text{ or } \varphi_g = H^{-1}\varphi'_g H.$$

**Definition 7.** (see [14]) Given a group  $G$  and an element  $g \in G$ , let  $A_g = A(\mathcal{C}(G, \{g\}))$ , the adjacent matrix of Cayley graph  $\mathcal{C}(G, \{g\})$ . Additionally, given

a group  $G$  and a multiset  $S$  of elements of  $G$ , let  $A_S = A(\mathcal{C}(G, S))$ . Then we can define the map  $\psi : G \rightarrow GL_{|G|}(\mathbb{C})$  given by  $g \rightarrow A_g$ .

**Definition 8.** (see [14]) Let  $R_h$  be  $h \times h$  matrix with entries 1 in positions  $(1, 2), (2, 3), \dots, (h-1, h), (h, 1)$ . Let  $\Omega_h = \text{diag}\{e^{2\pi i \frac{hk^0}{n}}, e^{2\pi i \frac{hk^1}{n}}, \dots, e^{2\pi i \frac{hk^{m-1}}{n}}\}$  be  $m \times m$  matrix satisfy  $k^m \equiv 1 \pmod{n}$ . Let  $X = \text{diag}\{\Omega_1, \Omega_2, \dots, \Omega_n\}$  and  $Y = \text{diag}\{R_m, R_m, \dots, R_m\}$  be two  $n \times n$  block matrices with  $m \times m$  blocks. We can also define the map  $\bar{\psi} : C_n \rtimes_{\theta(k)} C_m \rightarrow GL_{nm}(\mathbb{C})$  given by  $x^a y^b \rightarrow X^a Y^b$ .

**Lemma 17.** (see [14])  $\psi$  and  $\bar{\psi}$  are two representations of  $C_n \rtimes_{\theta(k)} C_m$  in  $GL_{nm}(\mathbb{C})$  and these two representations of  $C_n \rtimes_{\theta(k)} C_m$  are isomorphic group representations.

**Remark 4.** (see [14]) Now we know that these representations are isomorphic, then there exists an invertible matrix  $H$  such that for each  $g \in G$

$$\psi_g = H^{-1} \bar{\psi}_g H.$$

So  $\psi_g$  and  $\bar{\psi}_g$  will have the same characteristic polynomial and minimum polynomial.

**Lemma 18.** (see [14]) Given a group  $G$  and a multiset  $S$  of elements of  $G$ , then

$$A_S = \sum_{s \in S} A_s.$$

The symbol  $\chi(A, \gamma)$  and  $\chi_0(A, \gamma)$  are denoted with characteristic polynomial and minimum polynomial of  $A$  respectively.

**Lemma 19.** Let matrix  $A = \text{diag}\{a_1, a_2, \dots, a_n\} J_n$ , then the minimum polynomial  $\chi_0(A, \gamma)$  of  $A$  is a factor of  $\gamma(\gamma - \sum_{i=1}^n a_i)$ . Particularly, if  $\sum_{i=1}^n a_i \neq 0$ , then  $\chi_0(A, \gamma) = \gamma(\gamma - \sum_{i=1}^n a_i)$ .

*Proof.* Let  $s = \sum_{i=1}^n a_i$ , then

$$A(A - sI) = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} J_n \left( \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} J_n - sI \right) = 0.$$

Thus  $\chi_0(A, \gamma) | \gamma(\gamma - \sum_{i=1}^n a_i)$ . Suppose  $\sum_{i=1}^n a_i \neq 0$ , then  $A \neq 0$  and  $A \neq \left(\sum_{i=1}^n a_i\right) I$ , so  $\chi_0(A, \gamma) = \gamma(\gamma - \sum_{i=1}^n a_i)$ .  $\square$

**Definition 9.** Let  $S_u = \sum_{h=0}^{m-1} \sum_{a \in H} e^{2\pi i \frac{uak^h}{n}}$  and  $E_u(h) = \sum_{a \in H} e^{2\pi i \frac{uak^h}{n}}$  for  $0 \leq u \leq n-1$ ,  $0 \leq h \leq m-1$ . Then  $S_0 = vm$ .

In [14], Nathan Foxa gave an expression of characteristic polynomial of the semidirect product of two cyclic groups and this result will be represented in the following Lemma.

**Lemma 20.** (see [14]) *The characteristic polynomial of the semidirect product of two cyclic groups is given by the following:*

$$\chi(A(\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, S)), \gamma) = \prod_{u=0}^{n-1} \chi\left(\sum_{x^a y^b \in S} (\Omega_{ua})(R_m)^b, \gamma\right).$$

Particularly, let  $S = A' \times C_m$ , then

$$\chi(A(\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m)), \gamma) = \gamma^{n(m-1)}(\gamma - vm) \prod_{u=1}^{n-1} (\gamma - S_u).$$

*Proof.* The first assertion is from [14]. To prove the second assertion, we note that if  $S = A' \times C_m$ , then

$$\begin{aligned} \sum_{x^a y^b \in S} (\Omega_u)^a (R_m)^b &= \sum_{a \in H} (\Omega_{ua}) \sum_{b=0}^{m-1} (R_m)^b = \sum_{a \in H} (\Omega_{ua}) J \\ &= \sum_{a \in H} \text{diag}\{e^{2\pi i \frac{uak^0}{n}}, e^{2\pi i \frac{uak^1}{n}}, \dots, e^{2\pi i \frac{uak^{m-1}}{n}}\} J \\ &= \text{diag}\{E_u(0), E_u(1), \dots, E_u(m-1)\} J. \end{aligned} \tag{10}$$

Then

$$\begin{aligned}\chi\left(\sum_{x^a y^b \in S} (\Omega_{ua})(R_m)^b, \gamma\right) &= |\gamma I - \sum_{a \in H} (\Omega_{ua})J| = |\gamma I - \sum_{a \in H} (\Omega_{ua})\mathbf{1}_m \mathbf{1}_m^T| \\ &= \gamma^{m-1}(\gamma - \mathbf{1}_m^T \sum_{a \in H} (\Omega_{ua})\mathbf{1}_m) = \gamma^{m-1}(\gamma - S_u).\end{aligned}$$

Thus

$$\chi(A(\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, S)), \gamma) = \prod_{u=0}^{n-1} \chi\left(\sum_{x^a y^b \in S} (\Omega_{ua})(R_m)^b, \gamma\right) = \prod_{u=0}^{n-1} \gamma^{m-1}(\gamma - S_u).$$

Then the result follows.  $\square$

In a similar way, we will give an expression of minimum polynomial of the semidirect product of two cyclic groups.

Let  $\mathbf{lcm}\{f_1(\gamma), f_2(\gamma), \dots, f_{n-1}(\gamma), f_n(\gamma)\} = \mathbf{lcm}\{f_u(\gamma) | 1 \leq u \leq n\}$  denote with the least common multiple polynomial among the following polynomials  $f_1(\gamma), f_2(\gamma), \dots, f_{n-1}(\gamma), f_n(\gamma)$ .

**Lemma 21.** *The minimum polynomial of the semidirect product of two cyclic groups is given by the following:*

$$\chi_0(A(\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m), \gamma) \Big| \mathbf{lcm}\{\gamma(\gamma - S_u) | 0 \leq u \leq n-1\}.$$

Additionally, if  $S_u \neq 0$  for each  $1 \leq u \leq n-1$ , then

$$\chi_0(A(\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m), \gamma) = \mathbf{lcm}\{\gamma(\gamma - S_u) | 0 \leq u \leq n-1\}.$$

*Proof.* Let  $G = C_n \rtimes_{\theta(k)} C_m$ ,  $S = A' \times C_m$ , Then from Lemma 18

$$\chi_0(A(\mathcal{C}(G, S)), \gamma) = \chi_0\left(\sum_{s \in S} A(\mathcal{C}(G, \{s\})), \gamma\right) = \chi_0\left(\sum_{s \in S} A_s, \gamma\right).$$

Since  $s \in G$ , it can be written uniquely as  $x^a y^b$  for some  $0 \leq a < n$  and some  $0 \leq b < m$ . From the Lemma 17 and Remark 4, we have

$$\chi_0\left(\sum_{x^a y^b \in S} A_{x^a y^b}, \gamma\right) = \chi_0\left(\sum_{x^a y^b \in S} X^a Y^b, \gamma\right).$$



From the definition of  $X$  and  $Y$  in Definition 8, we can obtain that the minimum polynomial of  $A(\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m))$  is

$$\mathbf{lcm}\{\chi_0(\sum_{x^a y^b \in S} (\Omega_u)^a (R_m)^b, \gamma) | 0 \leq u \leq n-1\}. \quad (11)$$

We can get

$$\chi_0(\sum_{x^a y^b \in S} (\Omega_u)^a (R_m)^b, \gamma) \Big| \gamma(\gamma - \sum_{h=0}^{m-1} \sum_{a \in H} e^{2\pi i \frac{u a k h}{n}}) = \gamma(\gamma - S_u)$$

from Lemma 19 and the equation (10) in Lemma 20. Thus from (11), we can obtain that

$$\chi_0(A(\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m)), \gamma) \Big| \mathbf{lcm}\{\gamma(\gamma - S_u) | 0 \leq u \leq n-1\}.$$

This proves the first assertion. Additionally, if  $S_u \neq 0$  for each  $1 \leq u \leq n-1$ , then from Lemma 19, we have

$$\chi_0(A(\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m)), \gamma) = \mathbf{lcm}\{\gamma(\gamma - S_u) | 0 \leq u \leq n-1\}.$$

The second assertion is proved.  $\square$

In the following theroem, we will give a sufficient and necessary condition of Cayley graph  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m)$  to be DSRG with parameters  $(nm, vm, \mu, \lambda, t)$  and propositions  $(\star)$   $t = \mu, \frac{vm}{\mu-\lambda} = n-1$  in term of  $S_1, S_2, \dots, S_{n-1}$ .

We note that both  $(pn, vn, \frac{n}{p-1}v^2, \frac{n}{p-1}v(v-1), \frac{n}{p-1}v^2)$ -DSRG and  $(mq, m-1, (m-1)/q, ((m-1)/q)-1, (m-1)/q)$ -DSRG satisfy propositions  $(\star)$ .

The  $r, s, \rho, \sigma$  occuring in the following Theorems 22,23 are defined as Proposition 2.

**Theorem 22.** *Cayley graph  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m)$  is a DSRG with parameters  $(nm, vm, \mu, \lambda, t)$  and  $t = \mu, \frac{vm}{\mu-\lambda} = n-1$  if and only if  $S_1 = S_2 \cdots = S_{n-1}$  are negative integers.*

*Proof.* Let  $A = A(\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m))$ . If  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m)$  is a DSRG with parameters  $(nm, vm, \mu, \lambda, t)$  and  $t = \mu, \frac{vm}{\mu-\lambda} = n-1$ . Then

from Proposition 2, the three distinct eigenvalues of it are  $vm$ ,  $\rho = 0$  and  $\sigma = \lambda - \mu < 0$  with multiplicities  $1, r = \frac{-vm + (\mu - \lambda)(nm - 1)}{\mu - \lambda} = n(m - 1), s = \frac{vm + \rho(nm - 1)}{\rho - \sigma} = \frac{vm}{\mu - \lambda} = n - 1$  respectively. From the Lemma 20, the characteristic polynomial of  $A$  is  $(\gamma - vm)\gamma^{n(m-1)}(\gamma - \sigma)^{n-1} = \gamma^{n(m-1)}(\gamma - vm) \prod_{u=1}^{n-1} (\gamma - S_u)$ , so  $S_1 = S_2 \cdots = S_{n-1} = \sigma < 0$ .

On the other hand, suppose  $S_1 = S_2 \cdots = S_{n-1}$  are negative integer  $\bar{d}$ , then from the Lemma 21, the minimum polynomial of  $A$  is  $\mathbf{lcm}\{\gamma(\gamma - S_u) | 0 \leq u \leq n - 1\} = \gamma(\gamma - vm)(\gamma - \bar{d})$ , and  $AJ = JA = vm$ . Then  $A(A - vmI)(A - \bar{d}I) = 0$ . Let  $B = A(A - \bar{d}I)$ , so  $(A - vmI)B = AB - vmB = 0$ , i.e.  $AB = vmB$ , then each column of  $B$  is an eigenvector corresponding to simple eigenvalue  $vm$  (from the Perron-Frobenius theory, can see, e.g., Horn and Johnson[15]), but the eigenspace associated with the eigenvalue  $vm$  has dimension one and hence each column of  $B$  is a suitable multiple of  $\mathbf{1}_{nm}$ . Let  $B = (b_1 \mathbf{1}_{nm}, b_2 \mathbf{1}_{nm}, \dots, b_{nm} \mathbf{1}_{nm})$ , since  $\mathbf{1}_{nm}^T B = \mathbf{1}_{nm}^T A(A - \bar{d}I) = vm(vm - \bar{d}) \mathbf{1}_{nm}^T$ , we can get  $nmb_1 = nmb_2 = \dots = nmb_n = vm(vm - \bar{d})$ . Thus  $A(A - \bar{d}I) = B = \frac{v(vm - \bar{d})}{n} J \triangleq \mu J$ , then

$$A^2 = \frac{v(vm - \bar{d})}{n} J + \bar{d}A = \mu I + \lambda A + \mu(J - I - A).$$

where  $\mu + \bar{d} \triangleq \lambda$ . Thus Cayley graph  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m)$  is a DSRG with parameters  $(nm, vm, \mu, \lambda, t)$  and  $t = \mu$ . Additionally, from Lemma 20 the characteristic polynomial of  $A$  is  $\chi(A(\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m)), \gamma) = \gamma^{n(m-1)}(\gamma - vm) \prod_{u=1}^{n-1} (\gamma - \bar{d}) = \gamma^{n(m-1)}(\gamma - vm) \prod_{u=1}^{n-1} (\gamma - (\lambda - \mu))$ , hence  $\rho = 0$ ,  $\sigma = \lambda - \mu$ , and the multiplicity of eigenvalue  $\sigma = \lambda - \mu$  is  $s = n - 1$  which implies that  $s = \frac{vm + \rho(nm - 1)}{\rho - \sigma} = \frac{vm}{\mu - \lambda} = n - 1$ , then the result follows.  $\square$

**Remark 5.** We can verify Theorem 9 from Theorem 22, recalling the conditions in the Theorem 9, Cayley graph  $\mathcal{C}(C_p \rtimes_{\theta(m)} C_n, A' \times C_n)$  satisfies  $S_0 = vn$ , and

$$S_u = \sum_{h=0}^{n-1} \sum_{l \in H} e^{2\pi i \frac{ulm^h}{p}} = \sum_{h=0}^{n-1} \sum_{l \in H} e^{2\pi i \frac{ulm^h}{p}} = \frac{n}{p-1} \sum_{h=1}^{p-1} \sum_{l \in H} e^{2\pi i \frac{h}{p}} = -\frac{nv}{p-1}$$

for  $1 \leq u \leq p - 1$ , so  $\mathcal{C}(C_p \rtimes_{\theta(m)} C_n, A' \times C_n)$  is a DSRG with parameters  $(pn, vn, \mu, \lambda, t)$  such that  $t = \mu$ ,  $\lambda - \mu = -\frac{nv}{p-1}$ , and  $\mu = \frac{v(vn + \frac{nv}{p-1})}{p} = \frac{nv^2}{p-1}$ ,

i.e.  $(pn, vn, \frac{n}{p-1}v^2, \frac{n}{p-1}v(v-1), \frac{n}{p-1}v^2)$ .

A generalized result will be exhibited in the following Theorem.

**Theorem 23.** *Cayley graph  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m)$  is a DSRG with given  $s, \sigma < 0$  if and only if  $s$  numbers in  $S_1, S_2, \dots, S_{n-1}$  have same value  $\sigma$  and others are 0, and if  $S_u = 0$ , then  $E_u(h) = 0$  for  $0 \leq h \leq m-1$ .*

*Proof.* Let  $A = A(\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m))$ . Suppose  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m)$  is a DSRG with given  $s, \sigma < 0$ , then from Proposition 2 and Lemma 20, the characteristic polynomial of  $A$  is  $(\gamma - vm)(\gamma - \rho)^r(\gamma - \sigma)^s = \gamma^{n(m-1)}(\gamma - vm) \prod_{u=1}^{n-1} (\gamma - S_u)$ . Thus  $\rho = 0$ , and  $s$  numbers in  $S_1, S_2, \dots, S_{n-1}$  have same value  $\sigma$ . And if  $S_u = 0$ , we can obtain that  $\sum_{a \in H} (\Omega_{ua})J = 0$ . Otherwise,  $\chi_0(\sum_{a \in H} (\Omega_{ua})J, \gamma) = \gamma^2$ , then from lemma 21, the power of factor  $\gamma$  in  $\chi_0(A)$  greater than 1, this is a contradiction to  $\chi_0(A) = (\gamma - vm)\gamma(\gamma - \sigma)$ . Thus from equation (10), we can get  $E_u(h) = 0$  for each  $0 \leq h \leq m-1$ .

On the other hand, if  $S_1, S_2, \dots, S_{n-1}$  satisfy the conditions, then from the Lemma 21, the minimum polynomial of  $A$  is  $\text{lcm}\{\gamma(\gamma - vm), \gamma, \gamma - \sigma\} = (\gamma - vm)\gamma(\gamma - \sigma)$ . Similar to the proof of Theorem 22, we can also obtain that the Cayley graph  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m)$  is a DSRG. Additionally, from Lemma 20, the characteristic polynomial of  $A$  is  $\chi(A(\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m)), \gamma) = (\gamma - \sigma)^s(\gamma - vm)\gamma^r$ , then the result follows.  $\square$

**Remark 6.** If Cayley graph  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, A' \times C_m)$  is a DSRG with parameters  $(nm, vm, \mu, \lambda, t)$ , then  $t = \mu$ .

## 5.2. A sufficient and necessary condition of Cayley graph

$\mathcal{C}(C_n \rtimes_{\theta} C_m, (A' + e_A) \times C_m \setminus e_A)$  to be DSRG

When  $n$  is even, Hobart, Sylvia A., and T.Justin Shaw in [10] constructed Cayley graph  $\mathcal{C}(G, \widehat{S})$  with dihedral group  $G = D_n = \langle b, a | b^n = a^2 = e, ab = b^{-1}a \rangle$ , and  $\widehat{S} = \{b, b^2, \dots, b^{\frac{n}{2}-1}, a, ab, ab^2, \dots, ab^{\frac{n}{2}-1}\} = \{b^0, b^1, \dots, b^{\frac{n}{2}-1}\} \times \{a^0, a^1\} \setminus \{b^0\}$ . This Cayley graph is a DSRG with parameters  $(2n, n-1, \frac{n}{2}-1, \frac{n}{2}-1, \frac{n}{2})$ .

In this section, we will investigate  $\mathcal{C}(C_n \rtimes_{\theta} C_m, (A' + e_A) \times C_m \setminus e_A)$  respect to it. Let  $H$  be a proper subset of  $\{1, 2, \dots, n-1\}$  with  $|H| = v$ . And  $(A' + e_A) \times C_m \setminus e_A = \{x^a y^b, x^c | 1 \leq b \leq m-1, a \in H \cup \{0\}, c \in H\}$ .

**Definition 10.** Let  $S_u^* = \sum_{h=0}^{m-1} \sum_{a \in H \cup \{0\}} e^{2\pi i \frac{uak^h}{n}}$  and  $E_u^*(h) = \sum_{a \in H \cup \{0\}} e^{2\pi i \frac{uak^h}{n}}$  for  $0 \leq u \leq n-1, 0 \leq h \leq m-1$ . Then  $S_0^* = (v+1)m$ .

We can also obtain the characteristic polynomial and minimum polynomial of Cayley graph  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, (A' + e_A) \times C_m \setminus e_A)$  as section 5.1. Let  $S^* = (A' + e_A) \times C_m \setminus e_A$ .

**Lemma 24.** *The characteristic polynomial of the  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, S^*)$  is given by the following:*

$$\chi(A(\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, S^*)), \gamma) = (\gamma+1)^{n(m-1)} (\gamma+1 - (v+1)m) \prod_{u=1}^{n-1} (\gamma+1 - S_u^*).$$

*Proof.* We can obtain

$$\begin{aligned} \sum_{x^a y^b \in S^*} (\Omega_u)^a (R_m)^b &= \sum_{a \in H \cup \{0\}} (\Omega_{ua}) \sum_{b=0}^{m-1} (R_m)^b - \Omega_0 = \sum_{a \in H \cup \{0\}} (\Omega_{ua}) J - I \\ &= \sum_{a \in H \cup \{0\}} \text{diag}\{e^{2\pi i \frac{uak^0}{n}}, e^{2\pi i \frac{uak^1}{n}}, \dots, e^{2\pi i \frac{uak^{m-1}}{n}}\} J - I \\ &= \text{diag}\{E_u^*(0), E_u^*(1), \dots, E_u^*(m-1)\} J - I. \end{aligned} \quad (12)$$

and

$$\chi\left(\sum_{x^a y^b \in S^*} (\Omega_{ua}) (R_m)^b, \gamma\right) = |(\gamma+1)I - \sum_{a \in H \cup \{0\}} (\Omega_{ua}) \mathbf{1}_m \mathbf{1}_m^T| = (\gamma+1)^{m-1} (\gamma+1 - S_u^*).$$

From the Lemma 20, we have

$$\chi(A(\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, S^*)), \gamma) = \prod_{u=0}^{n-1} (\gamma+1)^{m-1} (\gamma+1 - S_u^*),$$

then the result follows.  $\square$

**Lemma 25.** *The minimum polynomial of  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, S^*)$  is given by the following:*

$$\chi_0(A(\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, S^*)), \gamma) \Big|_{\text{lcm}\{(\gamma+1)(\gamma+1 - S_u^*) | 0 \leq u \leq n-1\}}.$$

*Proof.* From the Lemma 19 and the equation (12) in Lemma 24, we have

$$\chi_0\left(\sum_{x^a y^b \in S^*} (\Omega_{ua})(R_m)^b, \gamma\right) = \chi_0\left(\sum_{a \in H \cup \{0\}} (\Omega_{ua})J - I, \gamma\right) = \chi_0\left(\sum_{a \in H \cup \{0\}} (\Omega_{ua})J, \gamma + 1\right),$$

then the result follows from Lemma 21 by replacing  $\gamma$  with  $\gamma + 1$ .  $\square$

**Theorem 26.** *Cayley graph  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, S^*)$  is a DSRG with given  $r, \rho$  if and only if  $r$  numbers in  $S_1^*, S_2^* \dots, S_{n-1}^*$  have same value  $1 + \rho$  and others are 0, and if  $S_u^* = 0$ , then  $E_u^*(h) = 0$  for  $0 \leq h \leq m - 1$ .*

*Proof.* Let  $A = A(\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, S^*))$ . Suppose Cayley graph  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, S^*)$  is a DSRG with given  $r, \rho$ . Then from Proposition 2 and Lemma 24, the characteristic polynomial of  $A$  is  $(\gamma + 1 - (v + 1)m)(\gamma - \rho)^r(\gamma - \sigma)^s = \prod_{u=0}^{n-1} (\gamma + 1)^{m-1}(\gamma + 1 - S_u)$ . Thus  $\sigma = -1$ , and  $r$  numbers in  $S_1^*, S_2^* \dots, S_{n-1}^*$  has same value  $1 + \rho$ . We note that  $\chi_0(A) = (\gamma + 1 - vm)(\gamma + 1)(\gamma - \rho)$ , which imply that if  $S_u^* = 0$ , then  $\sum_{a \in H \cup \{0\}} (\Omega_{ua})J = 0$ . Thus from equation (12), we can get  $E_u^*(h) = 0$  for all  $0 \leq h \leq m - 1$ .

On the other hand, suppose  $S_1^*, S_2^* \dots, S_{n-1}^*$  satisfy the conditions, then from the Lemma 25, the minimum polynomial of  $A$  is  $\text{lcm}\{\gamma(\gamma + 1 - (v + 1)m), \gamma + 1, \gamma - \rho\} = (\gamma + 1 - (v + 1)m)(\gamma + 1)(\gamma - \rho)$ . Similar to the proof of Theorem 22, the Cayley graph  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, S^*)$  is a DSRG. Additionally, the characteristic polynomial of  $A$  is  $\chi(A(\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, S^*)), \gamma) = (\gamma + 1)^s(\gamma + 1 - (v + 1)m)(\gamma - \rho)^r$ , then the result follows.  $\square$

**Remark 7.** (1). If Cayley graph  $\mathcal{C}(C_n \rtimes_{\theta(k)} C_m, S^*)$  is a DSRG with parameters  $(nm, m(v + 1) - 1, \mu, \lambda, t)$ , then  $t = \lambda + 1$ . Indeed, from Proposition 1,  $\sigma = -1$  implies that  $(\lambda - \mu + 2)^2 = d = (\lambda - \mu)^2 + 4(t - \mu)$ , i.e.  $t = \lambda + 1$ .

(2). Using Theorem 26, we can verify  $\mathcal{C}(D_n, \widehat{S})$  constructed by Hobart, Sylvia A., and T. Justin Shaw is a DSRG with parameters  $(2n, n - 1, \frac{n}{2} - 1, \frac{n}{2} - 1, \frac{n}{2})$  easily. Indeed,  $\mathcal{C}(D_n, \widehat{S})$  satisfies  $S_0 = 2 \times \frac{n}{2} = n$ , and  $S_u^* = \sum_{h=0}^1 \sum_{a=0}^{\frac{n}{2}-1} e^{2\pi i \frac{ua(-1)^h}{n}} =$

$\sum_{a=0}^{n-1} e^{2\pi i \frac{ua}{n}} + 1 - e^{\pi i u} = 1 - (-1)^u$ . Thus

$$S_u^* = \begin{cases} 2, & \text{if } 2|u \\ 0, & \text{if } 2 \nmid u \end{cases}$$

and if  $S_u^* = 0$ , then  $2|u$ , so  $E_u^*(h) = \sum_{a=0}^{\frac{n}{2}-1} e^{2\pi i \frac{u(-1)^h a}{n}} = \frac{1 - e^{2\pi i \frac{u \frac{n}{2} (-1)^h}{n}}}{1 - e^{2\pi i \frac{u(-1)^h}{n}}} = 0$ . From Theorem 26, the Cayley graph  $\mathcal{C}(D_n, S)$  is a DSRG with  $r = \frac{n}{2}$ ,  $\rho = 1$ ,  $\sigma = -1$ . Thus the parameters of  $\mathcal{C}(D_n, S)$  is  $(2n, n-1, \frac{n}{2}-1, \frac{n}{2}-1, \frac{n}{2})$ .

(3). We can verify Theorem 10 from Theorem 26. Indeed, under the hypothesis of Theorem 10, the Cayley graph  $\mathcal{C}(C_p \rtimes_{\theta(m)} C_n, (A' + e_A) \times C_m \setminus e_A)$  satisfies

$$S_u^* = \sum_{h=0}^{n-1} \sum_{l \in H \cup \{0\}} e^{2\pi i \frac{u l m h}{p}} = \frac{n}{p-1} \sum_{h=1}^{p-1} \sum_{l \in H} e^{2\pi i \frac{h}{p}} + n = n - \frac{nv}{p-1}$$

for each  $1 \leq u \leq p-1$ , so Cayley graph  $\mathcal{C}(C_p \rtimes_{\theta(m)} C_n, (A' + e_A) \times C_m \setminus e_A)$  is a DSRG with  $\rho = n-1 - \frac{nv}{p-1}$ ,  $r = n-1$ , and  $\sigma = -1$ . Thus the parameters of it is  $(pn, n(v+1)-1, \frac{nv(v+1)}{p-1}, n-2 + \frac{nv^2}{p-1}, n-1 + \frac{nv^2}{p-1})$ .

## 6. The out(in)-neighbour set and automorphsim group of DSRG (Cayley graphs)

### 6.1. The out(in)-neighbour set of DSRG(Cayley graphs)

In this section, we make a discussion of the vertices which have the same out-neighbour set (or in-neighbour set), the following Lemma gives an upper bound of the number of these vertices. Throughout this section, digraph  $D$  is directed strongly regular graph with parameters  $(n, k, \mu, \lambda, t)$ .

**Lemma 27.** (see [12]) *If a DSRG with parameters  $(n, k, \mu, \lambda, t)$  has a set  $S$  of vertices, all of which have the same set  $N$  of out-neighbour set (or they all have the same set of in-neighbour set), then  $|S| \leq k - \lambda$ ,  $|S| \leq n - 2k + t$ .*

**Definition 11.** Let  $S_1, S_2, \dots, S_t$  be the partition of  $V(D)$  such that any two vertices from the same  $S_i$  have the same out-neighbour set and any two vertices from distinct  $S_i$  have distinct out-neighbour set. We denote this partition with

$P_{out}(D) = \{S_1, S_2, \dots, S_t\}$ . For in-neighbour set, we can also define  $P_{in}(D)$  as above.

From Lemma 27, we can obtain that  $|S_i| \leq \min\{k - \lambda, n - 2k + t\}$  for all  $0 \leq i \leq t$ . The following Lemma gives an improvement of Lemma 27, and this Lemma gives an upper bound of  $|S_i| + |S_j|$  for distinct  $i$  and  $j$ .

**Lemma 28.** *If  $D$  a directed strongly regular graph with parameters  $(n, k, \mu, \lambda, t)$ , and  $\lambda > 0$ . Then for distinct  $0 \leq i, j \leq t$ ,  $|S_i| + |S_j| \leq \max\{k, n - 2k + 2\beta\}$ , where  $\beta = \min\{k - \lambda, \mu, n - 2k + t\}$ .*

*Proof.* We can assume  $i = 1$  and  $j = 2$ . At first, we claim that  $N_D^+(S_2) \not\subseteq S_1$ ,  $N_D^+(S_1) \not\subseteq S_2$ , as  $|S_1| \leq k - \lambda < k = |N_D^+(S_2)|$ ,  $|S_2| \leq k - \lambda < k = |N_D^+(S_1)|$  from Lemma 27. Since  $D$  is loopless, we have  $S_1 \cap S_2 = \emptyset$ ,  $S_1 \cap N_D^+(S_1) = S_2 \cap N_D^+(S_2) = \emptyset$

**Case 1.**  $|S_1| + |S_2| \leq k$ .

**Case 2.** If  $|S_1| + |S_2| > k$ , then  $N_D^+(S_1) \cap N_D^+(S_2) = \emptyset$ . Thus  $|S_1| + |S_2| \leq n + |S_1 \cap N_D^+(S_2)| + |S_2 \cap N_D^+(S_1)| - 2k$ . Let  $u = |S_2 \cap N_D^+(S_1)| \leq |S_2| \leq n - 2k + t$ ,  $z = |S_1 \cap N_D^+(S_2)| \leq |S_1| \leq n - 2k + t$ , then  $|S_1| + |S_2| \leq n - 2k + u + z$ .

**Case 2.1.**  $S_1 \not\subseteq N_D^+(S_2)$ ,  $S_2 \not\subseteq N_D^+(S_1)$ . See the Figure 1.

Let  $v \in S_1 \setminus N_D^+(S_2)$ ,  $w \in N_D^+(S_2) \setminus S_1$ ,  $x \in S_2 \setminus N_D^+(S_1)$ ,  $y \in N_D^+(S_1) \setminus S_2$ . Since  $v \not\rightarrow w$ , we can obtain that  $S_2 \cap N_D^+(S_1) \subseteq N_D^+(v) \cap N_D^-(w)$ , so  $u \leq \mu$ ; Since  $v \not\rightarrow x$ , then  $|N_D^+(S_1) \setminus S_2| \geq \mu$ , and  $u \leq k - \mu$ ; Since  $v \rightarrow y$ , then  $|N_D^+(S_1) \setminus S_2| \geq \lambda$ , and  $u \leq k - \lambda$ . Thus  $u \leq \min\{k - \mu, k - \lambda, \mu, n - 2k + t\} \triangleq \alpha$ . In a similar way,  $z \leq \alpha$ , then  $|S_1| + |S_2| \leq n - 2k + u + z = n - 2k + 2\alpha$ .

**Case 2.2.**  $S_1 \not\subseteq N_D^+(S_2)$ ,  $S_2 \subseteq N_D^+(S_1)$ . See the Figure 2.

We can also get  $z \leq \min\{k - \mu, k - \lambda, \mu, n - 2k + t\} \triangleq \alpha$  as **case 2.1**. To give an upper bound on  $u$ , let  $v \in S_1 \setminus N_D^+(S_2)$ ,  $w \in N_D^+(S_2) \setminus S_1$ , and  $y \in N_D^+(S_1) \setminus S_2$ . Similar to the discussion of **case 2.1**, we can also obtain that  $u \leq \min\{k - \lambda, \mu, n - 2k + t\} \triangleq \beta$ . Thus  $|S_1| + |S_2| \leq n - 2k + u + z = n - 2k + \alpha + \beta \leq n - 2k + 2\beta$ .

**Case 2.3.**  $S_2 \not\subseteq N_D^+(S_1)$ ,  $S_1 \subseteq N_D^+(S_2)$ . We also have  $|S_1| + |S_2| \leq n - 2k +$

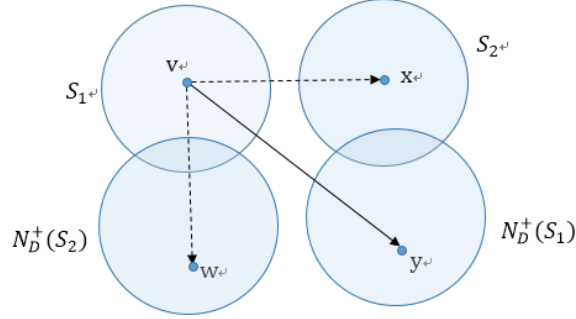


Figure 1:  $S_1 \not\subseteq N_D^+(S_2)$ ,  $S_2 \not\subseteq N_D^+(S_1)$

$$u + z = n - 2k + \alpha + \beta \leq n - 2k + 2\beta.$$

**Case 2.4.**  $S_2 \subseteq N_D^+(S_1)$ ,  $S_1 \subseteq N_D^+(S_2)$ . We also have  $|S_1| + |S_2| \leq n - 2k + 2\beta$ .

Base on our discussion, the result follows.

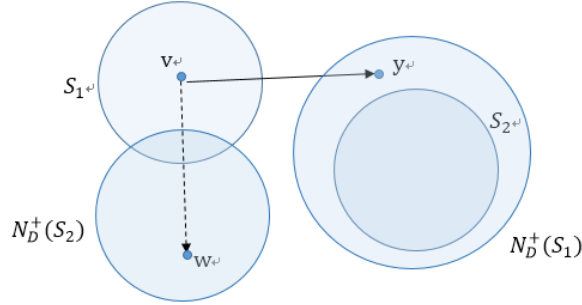


Figure 2:  $S_1 \not\subseteq N_D^+(S_2)$ ,  $S_2 \subseteq N_D^+(S_1)$

□

**Theorem 29.** Let  $D$  be a directed strongly regular Cayley graph  $\mathcal{C}(G, S)$  with parameters  $(n, k, \mu, \lambda, t)$ . Let  $G_S = \{g | g \in G, gS = S\}$  be a subgroup of  $G$ , and  $G = \bigcup_{i=1}^q a_i G_S$  to be the left coset decomposition of  $G$  with respect to  $G_S$ , where  $q = |G : G_S|$  is the index of  $G_S$  in  $G$ . Then we have

$$(a) \ P_{out}(D) = \{a_1 G_S, a_2 G_S, \dots, a_q G_S\};$$



- (b)  $|a_i G_S| = |G_S| \leq \min\{k - \lambda, n - 2k + t\}$ ;
- (c)  $|G_S|$  is a factor of  $\gcd(n, k, \lambda, \mu, t)$ ;
- (d)  $G_S = \{1_G\}$  if  $t \neq \mu$  or  $\gcd(n, k, \lambda, \mu, t) = 1$ .

*Proof.* We note that for distinct vertices  $x, y \in G$ , the out-neighbour set of  $x, y$  are  $xS, yS$  respectively, then  $xS = yS \Leftrightarrow x^{-1}y \in G_S \Leftrightarrow xG_S = yG_S$ . Hence  $P_{out}(D) = \{a_1 G_S, a_2 G_S, \dots, a_q G_S\}$ , proving (a). The assertion (b) follows from Lemma 27 and the assertion (c) can be directly deduced by the Definition 11. Finally, if  $t \neq \mu$ , then for each  $g \in G_S$ , we multiply  $g$  on the left side of equation (7), we can get

$$\bar{S}^2 = tg + \lambda \bar{S} + \mu(\bar{G} - g - \bar{S}) = (t - \mu)g + (\lambda - \mu)\bar{S} + \mu\bar{G}.$$

Note that  $\bar{S}^2 = (t - \mu)e + (\lambda - \mu)\bar{S} + \mu\bar{G}$ , so  $g = e$  and  $G_S = \{e_G\}$ . On the other hand, if  $\gcd(n, k, \lambda, \mu, t) = 1$ , then from assertion (c), we can also obtain that  $|G_S| = 1$ ,  $G_S = \{e_G\}$ , proving (d).  $\square$

Let  $S^{-1} = \{a^{-1} | a \in S\}$ , then we can also obtain the following theorem as above.

**Theorem 30.** Let  $D$  be a directed strongly regular Cayley graph  $\mathcal{C}(G, S)$  with parameters  $(n, k, \mu, \lambda, t)$ . Let  $G_{S^{-1}} = \{g | g \in G, gS^{-1} = S^{-1}\}$ , and  $G = \bigcup_{i=1}^{q'} a_i G_{S^{-1}}$  to be the left coset decomposition of  $G$  with respect to  $G_{S^{-1}}$ , where  $q' = |G : G_{S^{-1}}|$  is the index of  $G_{S^{-1}}$  in  $G$ . Then we have

- (a)  $P_{in}(D) = \{a'_1 G_{S^{-1}}, a'_2 G_{S^{-1}}, \dots, a'_{q'} G_{S^{-1}}\}$ ;
- (b)  $|a_i G_{S^{-1}}| = |G_{S^{-1}}| \leq \min\{k - \lambda, n - 2k + t\}$ ;
- (c)  $|G_{S^{-1}}|$  is a factor of  $\gcd(n, k, \lambda, \mu, t)$ ;
- (d)  $G_{S^{-1}} = \{1_G\}$  if  $\gcd(n, k, \lambda, \mu, t) = 1$ .

*Proof.* We note that for distinct vertices  $x, y \in G$ , the in-neighbour set of  $x, y$  are  $xS^{-1}, yS^{-1}$  respectively, then  $xS^{-1} = yS^{-1} \Leftrightarrow x^{-1}y \in G_{S^{-1}} \Leftrightarrow xG_{S^{-1}} = yG_{S^{-1}}$ . Then  $P_{in}(D) = \{a'_1 G_{S^{-1}}, a'_2 G_{S^{-1}}, \dots, a'_{q'} G_{S^{-1}}\}$ , proving (a). The assertion (b) follows from Lemma 27, and the assertion (c) can be directly deduced by the Definition 11. Finally, from assertion (c), we can also obtain that  $|G_S| = 1$ ,  $G_S = \{e_G\}$ , proving (d).  $\square$

**Definition 12.** Let  $\mathcal{C}(G, S)$  be a DSRG. The digraph  $\mathcal{S}_{out}(\mathcal{C}(G, S))$  is defined by  $V(\mathcal{S}_{out}(\mathcal{C}(G, S))) = [G : G_S] = \{a_1 G_S, a_2 G_S, \dots, a_q G_S\}$ , and  $a_i G_S \rightarrow a_j G_S$  if only if  $a_i^{-1} a_j \in S$ , i.e.  $E(\mathcal{S}_{out}(\mathcal{C}(G, S))) = \{(a_i G_S, a_j G_S) | (a_i, a_j) \in \mathcal{C}(G, S)\}$ .

We can also define digraph  $\mathcal{S}_{in}(\mathcal{C}(G, S))$  with  $V(\mathcal{S}_{in}(\mathcal{C}(G, S))) = [G : G_{S^{-1}}] = \{a'_1 G_{S^{-1}}, a'_2 G_{S^{-1}}, \dots, a'_{q'} G_{S^{-1}}\}$  and  $E(\mathcal{S}_{in}(\mathcal{C}(G, S))) = \{(a_i G_{S^{-1}}, a_j G_{S^{-1}}) | (a_i, a_j) \in \mathcal{C}(G, S)\}$ .

**Theorem 31.** Let  $\mathcal{C}(G, S)$  be a directed strongly regular Cayley graph with parameters  $(n, k, \mu, \lambda, t)$ , then digraphs

- (1)  $\mathcal{S}_{out}(\mathcal{C}(G, S))$  is a  $(\frac{n}{|G_S|}, \frac{k}{|G_S|}, \frac{\mu}{|G_S|}, \frac{\lambda}{|G_S|}, \frac{t}{|G_S|})$ -DSRG;
- (2)  $\mathcal{S}_{in}(\mathcal{C}(G, S))$  is a  $(\frac{n}{|G_{S^{-1}}|}, \frac{k}{|G_{S^{-1}}|}, \frac{\mu}{|G_{S^{-1}}|}, \frac{\lambda}{|G_{S^{-1}}|}, \frac{t}{|G_{S^{-1}}|})$ -DSRG.

*Proof.* We first prove the assertion for  $\mathcal{S}_{out}(\mathcal{C}(G, S))$ . From the definition of  $\mathcal{S}_{out}(\mathcal{C}(G, S))$ , we have  $|V(\mathcal{S}_{out}(\mathcal{C}(G, S)))| = \frac{n}{|G_S|}$ ,  $d_{\mathcal{S}_{out}(\mathcal{C}(G, S))}^-(a_i G_S) = \frac{k}{|G_S|}$ , and  $d_{\mathcal{S}_{out}(\mathcal{C}(G, S))}^+(a_i G_S) \geq \frac{k}{|G_S|}$  for all  $1 \leq i \leq q$ . We note that

$$\sum_{i=1}^t d_{\mathcal{S}_{out}(\mathcal{C}(G, S))}^-(a_i G_S) = \sum_{i=1}^t d_{\mathcal{S}_{out}(\mathcal{C}(G, S))}^+(a_i G_S).$$

Then  $d_{\mathcal{S}_{out}(\mathcal{C}(G, S))}^-(a_i G_S) = d_{\mathcal{S}_{out}(\mathcal{C}(G, S))}^+(a_i G_S) = \frac{k}{|G_S|}$ , so  $\mathcal{S}_{out}(\mathcal{C}(G, S))$  is regular digraph.

If  $a_i G_S \rightarrow a_j G_S$ , then  $a_i \rightarrow a_j$  in  $\mathcal{C}(G, S)$ . Hence in  $\mathcal{C}(G, S)$ , the number of paths of length two from the vertex  $a_i$  to the vertex  $a_j$  is  $\lambda$ , so the number of paths of length two from the vertex  $a_i G_S$  to the vertex  $a_j G_S$  is  $\frac{\lambda}{|G_S|}$  in  $\mathcal{S}_{out}(\mathcal{C}(G, S))$ .

If  $a_i G_S \nrightarrow a_j G_S$ , then  $a_i \nrightarrow a_j$  in  $\mathcal{C}(G, S)$ . Hence in  $\mathcal{C}(G, S)$ , the number of paths of length two from the vertex  $a_i$  to the vertex  $a_j$  is  $\mu$ , then the number of paths of length two from the vertex  $a_i G_S$  to the vertex  $a_j G_S$  is  $\frac{\mu}{|G_S|}$  in  $\mathcal{S}_{out}(\mathcal{C}(G, S))$ . If  $a_i G_S = a_j G_S$ , then  $a_i G_S$  contains in the  $\frac{t}{|G_S|}$  2-cycle exactly. Thus  $\mathcal{S}_{out}(\mathcal{C}(G, S))$  is a DSRG with parameters  $(\frac{n}{|G_S|}, \frac{k}{|G_S|}, \frac{\mu}{|G_S|}, \frac{\lambda}{|G_S|}, \frac{t}{|G_S|})$ . The similar result for  $\mathcal{S}_{in}(\mathcal{C}(G, S))$  is also hold, since  $d_{\mathcal{S}_{in}(\mathcal{C}(G, S))}^+(a_i G_{S^{-1}}) = \frac{k}{|G_{S^{-1}}|}$ ,  $d_{\mathcal{S}_{in}(\mathcal{C}(G, S))}^-(a_i G_{S^{-1}}) \geq \frac{k}{|G_{S^{-1}}|}$  for each  $1 \leq i \leq q'$ .  $\square$

**Remark 8.** These digraphs  $\mathcal{S}_{out}(\mathcal{C}(G, S))$  and  $\mathcal{S}_{in}(\mathcal{C}(G, S))$  are different from Cayley coset graph.

### 6.2. The automorphism group of $DSRG(\text{Cayley graphs})$

In this section, we give an upper bound of  $|Aut(\mathcal{C}(G, S))|$  in term of  $|G_S|$  and  $|G_{S^{-1}}|$ .

**Theorem 32.** *Let  $D$  be a directed strongly regular Cayley graph  $\mathcal{C}(G, S)$  with parameters  $(n, k, \mu, \lambda, t)$ . Then*

$$|Aut(\mathcal{C}(G, S))| \leq \min\{(\frac{n}{|G_S|})! (|G_S|)!, (\frac{n}{|G_{S^{-1}}|})! (|G_{S^{-1}}|)!\}.$$

*Proof.* let  $\tau \in Aut(\mathcal{C}(G, S))$ , if  $x$  and  $y$  have the same out-neighbour set, then  $\tau(x)$  and  $\tau(y)$  also have the same out(in)-neighbour set. Thus for each  $1 \leq i \leq q$ ,  $\tau$  maps all vertices in  $a_i G_S$  to another coset  $a_{j_i} G_S$  for some  $j_i$ , where  $j_1, j_2, \dots, j_q$  is a permutation of  $1, 2, \dots, q$ . Thus  $|Aut(\mathcal{C}(G, S))| \leq (\frac{n}{|G_S|})! (|G_S|)!$ . In a similar way, we can also get  $|Aut(\mathcal{C}(G, S))| \leq (\frac{n}{|G_{S^{-1}}|})! (|G_{S^{-1}}|)!$ .  $\square$

### Acknowledgements

The author are grateful to those of you who support to us.

### References

### References

- [1] A. M. Duval, A directed graph version of strongly regular graphs, Journal of Combinatorial Theory, Series A 47 (1) (1988) 71–100.
- [2] J. A. Bondy, U. S. R. Murty, Graph theory with applications, Vol. 290, Citeseer, 1976.
- [3] F. Fiedler, M. Klin, C. Pech, Directed strongly regular graphs as elements of coherent algebras, General Algebra and Discrete Mathematics, Shaker Verlag, Aachen (1999) 69–87.

- [4] A. M. Duval, D. Iourinski, Semidirect product constructions of directed strongly regular graphs, *Journal of Combinatorial Theory, Series A* 104 (1) (2003) 157–167.
- [5] F. Fiedler, M. Klin, M. H. Muzychuk, Small vertex-transitive directed strongly regular graphs, *Discrete mathematics* 255 (1) (2002) 87–115.
- [6] M. Klin, A. Munemasa, M. Muzychuk, P.-H. Zieschang, Directed strongly regular graphs via coherent (cellular) algebras, preprint Kyushu-MPS-1997-12, Kyushu University.
- [7] F. Adams, A. Gendreau, O. Olmez, S. Y. Song, Construction of directed strongly regular graphs using block matrices, *arXiv preprint arXiv:1311.0494*.
- [8] O. Olmez, S. Y. Song, Construction of directed strongly regular graphs using finite incidence structures, *arXiv preprint arXiv:1006.5395*.
- [9] A. Brouwer, O. Olmez, S. Y. Song, Directed strongly regular graphs from designs, *European Journal of Combinatorics* 33 (6) (2012) 1174–1177.
- [10] S. A. Hobart, T. Justin Shaw, A note on a family of directed strongly regular graphs, *European Journal of Combinatorics* 20 (8) (1999) 819–820.
- [11] R. Feng, L. Zeng, Construction of directed strongly regular graphs as generalized cayley graphs, *arXiv preprint arXiv:1410.1161*.
- [12] L. K. Jørgensen, Directed strongly regular graphs with rank 5, *Linear Algebra and Its Applications* 477 (2015) 102–111.
- [13] J. Ma, B. Zhang, J. Liu, J. Zhao, Directed strongly regular graphs with rank 6, *Discrete Mathematics*.
- [14] N. Foxa, Spectra of semidirect products of cyclic groups, *Rose-Hulman Undergraduate Mathematics Journal* 11 (2).
- [15] R. A. Horn, C. R. Johnson, *Matrix analysis*, Cambridge university press, 2012.